§1. Definitions. Examples

$A \times A \rightarrow A$

Def 1. L is a lector space over $\mathbb{F}$. with a bilinear operation

$$L \times L \rightarrow L$$

$$(a,b) \longrightarrow [a,b]$$

Satisfy:

① bilinear

② $[x,x]=0 \Rightarrow [x,y]=-[y,x]$

☆ If char $\mathbb{F} = 2$

$$[x,y] = -[y,x]$$

$$\not\Downarrow$$

$$[x,x] = 0$$

☆ ③. Jacobian Identity.

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0$$

Example

(1) $\dim L = 1$

$\Rightarrow L = \mathbb{F}x$

$[x,x]=0$ unique!

(2) $\dim \mathcal{L} = 2$,

$$\mathcal{L} = \mathbb{F}x \oplus \mathbb{F}y$$

$$[x, y] = ax + by$$

① $ax + by = 0 \Rightarrow \forall u, v, [u, v] = 0$

② Assume $a \neq 0$.

$$[y, ax+by] = -a(ax+by)$$

Let $z = -\frac{y}{a}$, $w = ax+by$

$$\Rightarrow \quad [z, w] = w$$

$$L = \mathbb{F}z \oplus \mathbb{F}w$$

Homomorphism.

Def. $\phi : L_1 \to L_2$, linear. map

$$\phi([a,b]) = [\phi(a), \phi(b)]$$

Then ϕ is called a homomorphism.

If ϕ is invertible, it is called an isomorphism.

Def. L is a Lie algebra, $H \subseteq L$

is a subspace. $x, y \in H$, $[x, y] \in H$

$\Rightarrow$ H is called a "Lie" subspace.

(3). $\mathbb{F} = \mathbb{R}$, $L = \mathbb{R}^3$

$\underline{\underline{\alpha \times \beta}}$ is a Lie-algebra.

$$sl_2(\mathbb{R}) = \{ A \in M_2(\mathbb{R}) \mid tr A = 0 \}.$$

$$[A, B] = AB - BA$$

$$sl_2(\mathbb{R}) = \mathbb{R}x \oplus \mathbb{R}h \oplus \mathbb{R}y$$

$$x = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$y = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}.$$

$\mathbb{R}^3 \xrightarrow{\sim} sl_2$?

$$\underline{[h, x] = 2x \quad [h, y] = -2y}.$$

Not iso.! | $\phi(h)\times\phi(x) = 2\phi(x)$.

Ado theorem. dim L is finite

$\Rightarrow \exists\, n$, s.t. $\exists\, \phi$ injection.

s.t. $\phi(L) < gl_n(\mathbb{F})$

§ 1.2. Linear Lie algebra.

Example. Suppose A is an

asso. algebra

(i.e. $(xy)z = x(yz)$)

$\mathcal{L} \overset{\Delta}{=} A$ as vector space

$[x,y] = xy - yx$ (commutator)

$\Rightarrow$ $(\mathcal{L}, [\,,\,])$ is a Lie alg.

$\mathcal{L} = A^{-}$

$M_n(\mathbb{F})$. $\Rightarrow$ $gl_n(\mathbb{F})$

$\dim = n^2$

General Linear Lie algebra.

$sl_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid \operatorname{tr} A = 0\}$.

subalgebra of $gl_n(\mathbb{F})$.

$gl_n(\mathbb{F}) = sl_n(\mathbb{F}) \oplus \mathbb{F} I_n$

$(\operatorname{char} \mathbb{F} \nmid n)$.

If V is a vector space

$\mathrm{End}(V)$. is an asso. alg

$\Rightarrow$ $gl(V) = \boxed{\mathrm{End}(V)}$ is a Lie alg.

What is $sl(V)$?

$$sl(V) = \mathrm{Span}\{fg - gf \mid \forall f, g \in gl(V)\}.$$

$$\dim sl(V) = (\dim V)^2 - 1$$

Any subalg. of general linear

Lie alg is called a linear

lie alg.

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

$V = \mathbb{R}[X]$

$F : V \to V$

$f(x) \to f'(x)$

$G : V \to V$

$f(x) \to xf(x)$

$FG(f(x)) = f(x) + xf'(x)$

$G \cdot F(f(x)) = xf'(x)$

$\Rightarrow [F, G] = Id_V$.

$\Rightarrow$ In $gl_n(F)$, $[A, B] \neq I_n$ ✓

In $gl(V)$, $[f, g] \neq Id$ ×

Thm.

If L is a finite simple

Lie alg. over $\mathbb{C}$, then L is isomorphic to one of Lie alg. of type

$sl_{n+1}(\mathbb{C})$

A_n. $n \geq 1$

B_n. $n \geq 2$

C_n, $n \geq 3$

D_n, $n \geq 4$

E_6, E_7, E_8, F_4, G_2

classical linear Lie alg

Example (A_l, B_l, C_l, D_l)

A_l : $\dim V = l+1$

$$\mathfrak{sl}(V) \xrightarrow{\sim} \mathfrak{sl}_{n+1}(\mathbb{F})$$

$$= \left(\bigoplus_{i=2}^{n} (e_{11} - e_{ii})\mathbb{F}\right) \oplus$$

$$\left(\bigoplus_{i \neq j} \mathbb{F} e_{ij}\right)$$

$$[\mathrm{diag}(a_1, \cdots, a_{l+1}), e_{ij}]$$

$$= (a_i - a_j) e_{ij}$$

$$K = \{ A \in M_n(\mathbb{F}) / K^T = -K \}.$$

$[A,B]^T = -[A,B]$

$K \leq gl_n(\mathbb{F})$

C_l: $sp_{2l}(F)$

$\dim V = 2l$ Basis $\{v_1 \sim v_{2l}\}$

Non degenerated symmetric form.

skew

$f: V \times V \to F$ $f(u,v) = -f(v,u)$

$$S = \begin{pmatrix} & I_l \\ -I_l & \end{pmatrix}$$

$f(u,v) = u^T S v$

$Sp(V) = \{ x \in End(V) \mid f(x(v), w)$

$= -f(v, x(w)),$

$\forall v, w \}$

Claim: $Sp(V)$ is a subalg. of

$gl(V)$

$f((xy - yx)v, w) = f(xyv - yxv, w)$

$= -f(yv, xw) + f(xv, yw)$

$= f(v, yxw) - f(v, xyw)$

$x \in Sp(V)$

$\Leftrightarrow x^T S = -SX$

$$x=\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

$$\Rightarrow \quad x=\begin{pmatrix} \overset{\square}{\underset{\sim}{m}} & \overset{\triangle}{n} \\ p^{\triangle} & -m^{t} \end{pmatrix} \qquad \begin{matrix} n^{T}=n \\ p^{T}=p \end{matrix}$$

$sp(V)$ is a subalg of $sl(V)$.

$$\dim sp(V) = l^2 + l(l+1)$$

$$= \underline{2l^2+l}$$

Claim: $sp_2(\mathbb{F}) \xrightarrow{\sim} sl_2(\mathbb{F})$. ✓.

B_l: $\dim V = 2l+1$

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & I_l \\ \vdots & & & \\ 0 & I_l & & 0 \end{pmatrix}$$

$$f(u,v) = u^T S v \qquad f(u,v) = f(v,u)$$

$$\mathcal{O}_{2l+1}(\mathbb{F}) = \mathcal{O}(V) = \{ x \in \mathrm{End}(V) \mid$$

$$f(x(v),w) = -f(v, x(w)) \}$$

$\Rightarrow$ $\mathcal{O}_{2l+1}(\mathbb{F})$ is a subalg. of

$gl(V)$

$$\underbrace{x^T S = -S x}$$

$$x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \end{pmatrix}$$

$$\begin{pmatrix} c_2 & p & q \end{pmatrix}$$

$$x^T = \begin{pmatrix} a & \overline{c_1}^T & \overline{c_2}^T \\ b_1^T & m^T & p^T \\ b_2^T & n^T & q^T \end{pmatrix}$$

$\Leftrightarrow$ $a = 0$ $\quad c_2^T = -b_1$, $\quad c_1^T = -b_2$

$p^T = -P^T$ $\quad m^T = -q$

$n^T = -n$.

$$\dim O_{2l+1}(F) = 2l + l^2 + l(l-1)$$

$$= 2l^2 + l$$

claim: $Sp_2(F) \cong sl_2(F) \xrightarrow{\sim} O_3(F)$

$$Sp_4(\mathbb{F}) \xrightarrow{\sim} O_5(\mathbb{F})$$

D_l: $\dim V = 2l$ $\quad S = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$

$$O_{2l}(\mathbb{F}) = \{ x \mid f(x(v), w) = -f(v, x(w)) \}$$

$$\dim O_{2l}(\mathbb{F}) = 2l^2 - l$$

Claim: $O_6(\mathbb{F}) \xrightarrow{\sim} sl_4(\mathbb{F})$

$O_4(\mathbb{F}) \xrightarrow{\sim} sl_2(\mathbb{F}) \oplus sl_2(\mathbb{F})$

Example.

$$K_n = \{ A \in M_n(\mathbb{F}) \mid A^T = A \}.$$

subalg of $gl_n(\mathbb{F})$

Example

(1) $\mathcal{L}$ is a Lie alg. H, K are subspaces of $\mathcal{L}$

$[H,K] \triangleq \text{Span}_{\mathbb{F}}\{[a,b] \mid a\in H, b\in K\}$.

$\Rightarrow [\mathcal{L},\mathcal{L}]$ is subalg of $\mathcal{L}$

(2) $\text{alt}_n(\mathbb{F}) = \{A=(a_{ij}) \mid a_{ij}=0, \forall i>j\}$.

upper triangular matrics.

(b) $n_n(\mathbb{F}) = \{ A=(a_{ij}) \mid a_{ij}=0, \forall i \geq j \}$.

strictly upper ~

(c) $\delta_n(\mathbb{F}) = \{ \mathrm{diag}(a_1, \cdots, a_n) \mid a_i \in \mathbb{F} \}$

$t_n(\mathbb{F}) = n_n(\mathbb{F}) \oplus \delta_n(\mathbb{F})$.

$[t_n(\mathbb{F}), t_n(\mathbb{F})] = n_n(\mathbb{F})$.

L is a Lie alg.

$x \in L$, define $ad_x : L \to L$

$y \to [x,y]$

adjoint

$ad_x \in End(L)$

Jacobian Identity

$\Leftrightarrow ad_x([y,z]) = [ad_x(y), z] - [y, ad_x(z)]$

§1.3. Derivation.

Define.

If a linear map $\delta: A \to A$ s.t.

$\delta(ab) = \delta(a)b + a\delta(b)$

Then δ is a derivation of A

$\mathrm{Der}(A) = \{ \delta \mid \delta \text{ is a derivation} \}.$

$\subseteq \mathrm{End}(A)$

Claim: $\mathrm{Der}(A)$ is a Lie subalg

of $\mathfrak{gl}(A)$

$$ad_x(yz) = ad_x(y)z + y\,ad_x(z)$$

$$\overset{xyz - yzx}{} \qquad \qquad \overset{\parallel}{(xy - yx z) + y(xz - zx)}$$

(1). $A = \mathbb{F}[x]$ $\mathrm{Der}(A)$.

$\delta \in \mathrm{Der}(A)$

$\delta(1) = \delta(1 \cdot 1) = 2\delta(1) = 0$.

$f(x) := \delta(x)$

$\delta(x^k) = k \; x^{k-1} f(x)$.

$\Rightarrow \; \delta(p(x)) = f(x) \cdot \frac{d}{dx}(p(x))$.

$\mathrm{Der}(A) = \left\{ f(x) \frac{d}{dx} \,\middle|\, f(x) \in \mathbb{F}[x] \right\}$

(2) $A = \mathbb{C}[x, x^{-1}] = \sum_{i=m}^{n} a_i x^i \quad m < n \in \mathbb{Z}$

$\delta(x) = f(x)$

$\Rightarrow \; \delta(x^k) = k f(x) x^{k-1}, \; k \geq 1$.

(3) $A = L$ is a Lie-algebra.

$\mathrm{Der}(L)$?

$ad_x \in \mathrm{Der}(L)$.

Jacobian Identity

$$\Rightarrow [[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

$$\Rightarrow ad_x\, ad_y (z) - ad_y\, ad_x (z) = ad_{[x, y]}(z)$$

$$\Rightarrow \underline{\underline{[ad_x, ad_y] = ad_{[x, y]}}}$$

$\mathrm{Inn}(L) = \{ \mathrm{ad}x \mid x \in L \} \subseteq \mathrm{Der}(L)$.

is a Lie subalg of $\mathrm{Der}(L)$.

This is called inner derivation.

$\mathrm{Der}(L) \setminus \mathrm{Inn}(L)$ is called outer derivation.

(*). $\mathfrak{s}_n(\mathbb{F}) \subseteq \mathfrak{t}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$.

$\exists\, h \in \mathfrak{s}_n(\mathbb{F}) \Rightarrow \mathrm{ad}_h = 0$ on $\mathfrak{s}_n(\mathbb{F})$.

but $\mathrm{ad}_h \neq 0$ on $\mathfrak{t}_n(\mathbb{F})$.

§1.4. Abstract Lie algebra.

① $\dim \mathcal{L} = 1$ abelian Lie alg.

② . If $\forall x, y \in \mathcal{L}$ $[x, y] = 0$

$\mathcal{L}$ is called abelian Lie alg.

③ $\dim \mathcal{L} < +\infty$ $\{x_i\}$ basis.

$$[x_i, x_j] = \sum_{k=1}^{n} C_{ij}^{k} x_k$$

<u>Bilinear</u>

$$C_{ij}^{k} = - C_{ij}^{k}, \quad C_{ii}^{k} = 0$$

$$\sum_{k} \left(C_{ij}^{k} C_{kl}^{m} + C_{jk}^{k} C_{ki}^{m} + C_{li}^{k} C_{ij}^{m} \right) = 0$$

Example

Maybe not asso.

A is an $\mathbb{F}$-alg, define

$$(a,b,c)=(ab)c-a(bc)$$

If $\forall a,b,c$, $(a,b,c)=(b,a,c)$

then A is called a left-symmetric algebra.

asso. algebra $\Rightarrow$ left-symmetric.

$L=A$ as vector space

Define $[a,b]=ab-bc$

Claim: L is a Lie alg.

$(a,b,c) = (b,a,c)$

$\Leftrightarrow (ab)c - a(bc) = (ba)c - b(ac)$

Thm.

If A is a LSA

$\Rightarrow (A, [\,,\,])$ is a Lie alg.

Then $A \ncong sl_n(\mathbb{C})$.

§ 2. Ideals and homomorphisms.

§ 2.1. Ideals.

Def. A subspace I of a Lie alg $\mathcal{L}$ is called an ideal if $[x, y] \in I$, $\forall x \in I, y \in \mathcal{L}$

Example.

(1) Center.

$$Z(L) = \{x \in \mathcal{L} \mid [x, y] = 0, \forall y \in \mathcal{L}\}$$

$$Z(gl_n(\mathbb{F})) = \mathbb{F} I_n$$

$\forall x \in Z(\mathcal{L})$ $ad_x = 0$.

(2) (derived alg) $[\mathcal{L}, \mathcal{L}]$ is an ideal of $\mathcal{L}$

(3) I, J are ideals

$\Rightarrow$ $I \oplus J$ is an ideal

$I \cap J$ ～

(4) $[I, J]$ is an ideal

$M \twoheadrightarrow M$. φ.

Define.

If $L = [L, L]$ then L is called a perfect L-alg.

① $\mathfrak{gl}_n(\mathbb{F})$ is not perfect.

② $\mathfrak{sl}_n(\mathbb{F})$ is perfect.

③ $\mathfrak{t}_n(\mathbb{F})$ is not

④ $\mathrm{Der}(\mathbb{F}[x])$.

$$[f\frac{d}{dx}, g\frac{d}{dx}] = \underset{1}{f}\underset{x}{g'}\frac{d}{dx} - \underset{x}{g}\underset{1}{f'}\frac{d}{dx}.$$

is perfect.

$$[x\frac{d}{dx}. x^{k+1}\frac{d}{dx}] = k x^{k+1}$$

⑤ $\mathrm{Der}(\mathbb{C}[x, x^{-1}])$ is

$A_\ell, B_\ell, C_\ell, D_\ell$ are perfect.

Def. (Simple Lie alg).

If $[L, L] \neq 0$, L has only 0,

L as ideals, then L is called simple.

not abelian

Example (1) $(\mathbb{R}^3, \times)$ is simple

(2) $sl_2(\mathbb{F})$ char $\mathbb{F} \neq 2$

is simple $\underset{\shortparallel}{L}$

Pf: $[sl_2(\mathbb{F}), sl_2(\mathbb{F})] \neq 0$.

$I \neq 0$ $I \trianglelefteq L$

$0 \neq u = ax + bh + cy \in \mathcal{L}$

$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \mathrm{ad}_h(u) \in I$

$\mathrm{ad}_h(x) = 2x$

$\mathrm{ad}_h(h) = 0$

$\mathrm{ad}_h(y) = -2y$

$\Leftrightarrow 2ax - 2cy \in I$

$\mathrm{ad}_h(\mathrm{ad}_h(u)) \in I$

$\Leftrightarrow 4ax + 4cy$

$\Rightarrow ax, bh, cy \in I$

$\Rightarrow$ x, h, or $y \in I$

$\Rightarrow$ $I = L$

(*) Adjoint of diagonal matrix has many eigenvector (e_{ij}).

$$\mathrm{ad}_h e_{ij} = (\varepsilon_i - \varepsilon_j) e_{ij}$$

$\Rightarrow$ $sl_2(\mathbb{F})$ is simple

$sl_2(\mathbb{F}) \not\cong (\mathbb{R}^3, \times)$

$SL_2(\mathbb{F})$ is simple, $\mathrm{char}\,\mathbb{F} \neq 2$.

Definition $I \triangleleft L$

L/I quotient space

$$[\bar{x}, \bar{y}] \triangleq \overline{[x, y]}$$

Claim: This gives a Lie algebra structure

$$\overline{[x_1, y_1]} - \overline{[x, y]}$$

$$= \overline{[x_1, y_1]} - \overline{[x_1, y]} + \overline{[x_1, y]} - \overline{[x, y]}$$

Definition. L, L' are Lie algebras

$$G = L \oplus L'$$

$x_1, x_2 \in L \quad y_1, y_2 \in L'$

$$[x_1 + y_1, x_2 + y_2] = [x_1, x_2] + [y_1, y_2]$$

Direct sum / product.

* $L, L' \hookrightarrow G = L \oplus L'$

L, L' are ideals.

Definition.

i) Normalizer

K is a subspace of L

$$N_L(K) = \{ x \in L \mid [x, k] \subseteq k \}$$

$xK - kx$

(Analogy of group theory).

$$\mathrm{ad}_{[x,y]} z = \mathrm{ad}_x \mathrm{ad}_y (z) - \mathrm{ad}_y \mathrm{ad}_x (z)$$

$\Rightarrow$ If $x, y \in N_L(k)$.

$[x,y] \in N_L(K)$, subalgebra.

If K is a subalgebra

$K \subseteq N_L(K) \Rightarrow K \trianglelefteq N_L(K)$

(4) X is a subset of L

$C_L(X) = \{x \in L \mid [x,y]=0, \forall y \in X\}$

$C_L(X)$ is called centrallizer of X in L

$C_L(X)$ subalgebra

$C_L(L) = Z(L)$

(5) K is a subalg of L, if

$N_{\mathcal{L}}(K)=K$, then K is a self-normalizing subalg of $\mathcal{L}$

Example. $\mathcal{L} = sl_2(\mathbb{C})$

$H = \mathbb{C}h \qquad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$C_{\mathcal{L}}(H) = N_{\mathcal{L}}(H) = \mathbb{C}h = H$$

$$C_{\mathcal{L}}(\mathbb{C}x) = \mathbb{C}x$$

$$N_{\mathcal{L}}(\mathbb{C}x) = \mathbb{C}x \oplus \mathbb{C}h$$

Def. $\mathcal{L}, \mathcal{L}'$ are Lie algebra

$\varphi : \mathcal{L} \to \mathcal{L}'$ is a linear transformation, if $\varphi([x,y]) = [\varphi(x), \varphi(y)]$,

then φ is called a homomorphism

$*$ $\ker\varphi = 0$, monomorphism

$*$ $\operatorname{im}\varphi = L'$, epimorphism

Remark.

(1) $\varphi : L \to L' \Rightarrow \ker\varphi \triangleleft L$

$\operatorname{Im}\varphi$ is a subalg of L'

(2) $I \triangleleft L$ $\quad \pi : L \to L/I$

$\ker\pi = I$.

Prop. (a) $\varphi : L \to L'$

$$\exists\ L/\ker\varphi \cong \operatorname{im}\varphi$$

$I \subseteq \ker\varphi$

There is a unique homo. s.t.

$$\begin{tikzcd}
L \arrow[r] \arrow[dr, bend right] & L' \\
 & L/I \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

(b) $I, J \triangleleft L$, $I \subseteq J$

$\Rightarrow J/I \triangleleft L/I$

(c) $I, J \triangleleft L$ $\quad I+J/I \cong J/I\cap J$

1 Def. L is a Lie alg.

V is a vector space

(V, φ) is a representation if

$\varphi: L \to gl(V)$

$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$.

Example.

L is a Lie alg.

$ad: L \to gL(L)$

$x \mapsto ad_x$

is a representation, is called the adjoint rep.

$$\mathrm{ad}_{[x,y]}(z) = \mathrm{ad}_x\, \mathrm{ad}_y(z) - \mathrm{ad}_y\, \mathrm{ad}_x(z)$$

$\mathrm{Ker}\, \mathrm{ad} = Z(L)$

If $\mathrm{Ker}\, \mathrm{ad} = 0$

$$L \xrightarrow{\sim} \mathrm{ad}\, L$$

(***) $$\mathrm{Inn}(L) = \{\mathrm{ad}_x \mid x \in L\} \subseteq \mathrm{Der}(L)$$

$\delta \in \mathrm{Der}(L)$,

$$[\delta, \mathrm{ad}_x](y) = \delta[x,y] - [x, \delta(y)]$$

$$= [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)]$$

$$= \mathrm{ad}_{\delta(x)}(y)$$

$[\delta, \mathrm{ad}_x]$

Automorphism.

Def. An automorphism of L is an iso. $\varphi: L \to L$

Example.

L is a linear Lie alg.

$L \subseteq gl(V)$

If $g \in GL(V)$, gLg^{-1}, then

$L \to L$

$x \to gxg^{-1}$ is an automorphism.

$L = sl(V)$

char $\mathbb{F} = 0$, If $(ad_x)^k = 0$

$k-1$

$$\exp(\mathrm{ad}\,x) = I + \mathrm{ad}\,x + \frac{\mathrm{ad}\,x^2}{2!} + \cdots + \frac{\mathrm{ad}\,x^{k-1}}{(k-1)!}$$

Claim: $\exp(\mathrm{ad}\,x) \in \mathrm{Aut}(L)$

☆ $$\delta^n([y,z]) = \sum_{i=0}^{n} \binom{n}{i} [\delta^i(y), \delta^{n-i}(z)]$$

Leibniz's rule. (by induction)

$$\Leftrightarrow \frac{\delta^n}{n!}([y,z]) = \sum_{i=1}^{n} \left[\frac{\delta^i}{i!}(y), \frac{\delta^{n-i}}{(n-i)!}(z)\right]$$

Prop. $\delta \in \mathrm{Der}(L)$, $\delta^k = 0$, then

$\exp(\delta) \in \mathrm{Aut}(L)$

$$\Rightarrow [\exp \delta(x), \exp \delta(y)]$$

$$= \sum_{i,j=0}^{k-1} \left[\frac{\delta^i(x)}{i!}, \frac{\delta^j(y)}{j!}\right]$$

$$= \sum_{n=0}^{2k-2} \left(\sum_{i=0}^{n} \left[\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(x)}{(n-i)!} \right] \right)$$

$$= \exp \delta [x, y]$$

Remark. δ, y nilpotent

$$\exp(\delta + y) = \exp(\delta) \exp(y)$$

Prop. $\mathrm{Int}(L) = \langle \exp(\mathrm{ad}\, x) \mid \mathrm{ad}\, x \text{ nilp.} \rangle$

$\subseteq \mathrm{Aut}(L)$

Inner automorphisms.

Moreover, $\mathrm{Int}(L) \triangleleft \mathrm{Aut}(L)$

$\forall \varphi \in \mathrm{Aut}(L)$, $\mathrm{ad}_x \in \mathrm{Inn}(L)$

$$\varphi \, \mathrm{ad}_x \varphi^{-1}(y) = \varphi([x, \varphi^{-1}(y)])$$

$$= [\varphi(x), y]$$

$$= \mathrm{ad}_{\varphi(x)}(y)$$

$$\Rightarrow \varphi \, \mathrm{ad}_x \varphi^{-1} = \mathrm{ad}_{\varphi(x)}$$

$$\Rightarrow \varphi \exp(\mathrm{ad}_x) \varphi^{-1} = \exp(\mathrm{ad}_{\varphi(x)})$$

Example.

In $gl(V)$

☆. x nilpotent ⇒ ad_x nilpotent

x diagonalable ⇒ ad_x ~

$f_A(B) = AB - BA$

Prop. $L \subseteq gl(V)$, $x \in L$ nilp.

$\Rightarrow \exp(x)\, y \exp(-x) = \exp(ad\, x)(y)$

pf: $ad_x = L_x + R_{-x}$

$\exp(ad\, x)(y) = e^{L_x + R_{-x}}(y)$

$$= e^{L_x} e^{R_{-x}} (y)$$

$$= \exp(x)\, y \exp(-x)$$

§3. solvable / nilpotent Lie algebra.

§3.1. Solvable.

L is a Lie algebra

$$\underset{\text{deritive alg.}}{\underset{|}{[L, L]}} \triangleleft L$$

$$L \triangleright \underset{L^{(1)}}{\underset{\shortparallel}{[L, L]}} \triangleright \underset{L^{(2)}}{\underset{\shortparallel}{[L_1, L_1]}} \triangleright \cdots$$

derived series

Def. if $L^{(n)} = 0$ for some n,

L is called solvable Lie Alg.

Example. (1) $\dim L = 1 \Rightarrow$ solvable

L Abelian $\Rightarrow$ ✓

(2) $\dim L = 2$

$[L, L] = 0$ or

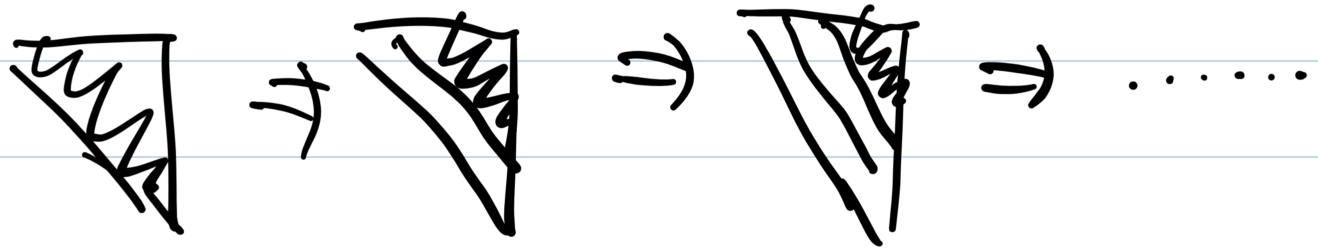
$L = \mathbb{F}x \oplus \mathbb{F}y \qquad [x, y] = y$

$\Rightarrow$ solvable.

(3) $\dim L = 3$

$t_n(\mathbb{F}) = \{(a_{ij}) \,/\, a_{ij} = 0, \forall i > j\}$

$(t_n(\mathbb{F}))^{(i)} = 0$, if $2^{i-1} \geq n$



Prop. L

• (a) If L is solvable $\Rightarrow$

$\forall$ subalg. $A \subseteq L$ is solvable.

• (b) $I \trianglelefteq L$, I, L/I solvable

$\Rightarrow L$ solvable.

•(c) I, J solvable $\Rightarrow$ $I+J$ solvable

$(I, J \triangleleft A)$.

(a): ✓

(b) $\Rightarrow$ (c): $I+J/I = J/I\cap J$.

(b): L/I is solvable

$\Rightarrow \exists\, k$, s.t. $(L/I)^{(k)} = 0$

$\Rightarrow L^{(k)} \subseteq I$

$\exists\, l$, s.t. $I^{(l)} = 0$

$\Rightarrow L^{(k+l)} = (L^{(k)})^{(l)} = 0$.

The converse part is trivial

Remark.

If $\dim L < +\infty$, L has a unique maximal solvable ideal

(finite Sum of solvable ideals is a solvable ideal).

Called the radical of L

$\mathrm{Rad}(L)$.

Def. If $\mathrm{Rad}(L) = 0$, L is called Semisimple.

If $\dim L < +\infty$

$\Rightarrow L/\mathrm{Rad}(L)$ is semisimple.

Example.

(1) L is simple $\Rightarrow$ L is semisimple.

$$\Downarrow$$

$$[L, L] = L \Rightarrow L^{(k)} = L$$

$\mathrm{Rad}(L) = 0$ or $\underset{\sim}{L}$ ✗

$\Rightarrow \mathrm{Rad}(L) = 0$

§ 3.2. Nilpotency

$$L^0 = L, \quad L^1 = [L, L] = L^{(1)}$$

$$L^2 = [L^1, L] \supseteq L^{(2)}$$

Lower central series/

descending central series

Def. $\mathcal{L}$ is nilpotent, if $\exists k$, $\mathcal{L}^k=0$.

Example.

(1) nilpotent $\Rightarrow$ solvable.

(2) $t_n(\mathbb{F})$ is solvable, but not nilpotent!

$[t_n(\mathbb{F}), t_n(\mathbb{F})] = n_n(\mathbb{F})$

$[t_n(\mathbb{F}), n_n(\mathbb{F})] = n_n(\mathbb{F})$.

Stable!

(3) $n_n(\mathbb{F})$ is nilpotent.

$$n_n(\mathbb{F})^k \subseteq \{\sum a_{ij} e_{ij} \mid j > i+k\}.$$

$$Z(n_n(\mathbb{F})) = \mathbb{F} e_{1n}.$$

Remark. Nilpotent Lie alg. has non-trivial centralizers.

$$L^k = 0 \Rightarrow L^{k-1} \subseteq Z(L).$$

Prop. (a) If L is nilpotent

$\Rightarrow$ subalgs, hom images, are nilp.

(b) (t_n/n_n, n_n are nilp., but t_n is not)

$$L/Z(L) \text{ is nilp}$$

$$\Uparrow$$

L is nilp.

(c) $L \neq 0$, nilp.

$\Rightarrow Z(L) \neq 0$.

Pf of (b):

$(L/Z(L))^n = 0$

$\Rightarrow L^n \subseteq Z(L)$

$\Rightarrow L^{n+1} = [L^n, L] = 0$.

Remark. $L^k = 0$

$\Rightarrow \mathrm{ad}_{x_1} \mathrm{ad}_{x_2} \cdots \mathrm{ad}_{x_k} = 0$

$\Rightarrow (ad_x)^k = 0 \ , \forall x \in L$

If ad_x is nilp., call x is ad-nilp.

Theorem (Engel)

If $\forall x \in L$, x is ad-nilp., then

L is nilpotent.

$ad: L \to gl(L) \qquad \ker ad = Z(L)$

$L/Z(L) \subseteq \underset{\sim}{gl(L)}.$

Lemma. $x \in gl(L)$

x is nilp. $\Rightarrow$ ad_x is nilp.

Remark.

$ad_x \in gl(gl(V))$ nilp.

x is nilp.

(Let $x = \mathrm{Id}$)

§ 3.3.

Pf of Engel's theorem.

Theorem. $L < gl(V)$, $\dim V < +\infty$. $\forall x \in L$, x nilp. and $V \neq 0$, then $\exists\, 0 \neq v \in V$, $L.v = 0$

Pf: Induction on $\dim L$.

$\dim L = 0, 1$ ✓

$\forall$ subalg K of L, define

$\varphi: K \to gl(L)$

$x \to ad_x$

$\varphi(K)$ acts on L nilpotently

$\Rightarrow$ K acts on L/K nilpotently

(K is subalg. $\Rightarrow$ this action is well defined).

By induction, $\exists\ 0 \neq \bar{y} \in L/K$, $\forall x \in K$

$\bar{\varphi}(x)(\bar{y}) = 0$

$\Leftrightarrow$ $[x, y] = 0$, $y \notin K$

$\Rightarrow$ $y \in N_K(L) \setminus K$

Take K is a maximal subalg.

of $\mathcal{L}$

$$K \subsetneq \underset{\substack{\uparrow \\ \text{subalg.}}}{N_{\mathcal{L}}(K)} \subseteq \mathcal{L}$$

$\Rightarrow N_{\mathcal{L}}(K) = \mathcal{L}$

$\Rightarrow K \trianglelefteq \mathcal{L}$

Claim: $\dim K = \dim \mathcal{L} - 1$

$\forall x \in \mathcal{L}/K$

$K + Fx$ is a subalg

$[K+Fx, K+Fx] = [K,K] + [K,x] \subseteq K$

$\square$

$\Rightarrow \mathcal{L} = K + Fz$

By induction,

$W = \{ v \in V \mid K.v = 0 \} \neq 0$

$K \trianglelefteq \mathcal{L}$

$\Rightarrow \forall\, v \in W, x \in \mathcal{L} \quad x.v \in W$

$\left(y(x.v) = [y,x](v) = 0, \forall\, y \in K \right)$

$\Rightarrow z.W \subseteq W$

z nilp. $\Rightarrow \exists \underset{\neq 0}{u} \in W, \; z.u = 0$

$\Rightarrow \mathcal{L}.u = 0$

Proof of Engel's theorem.

L is a Lie alg.

$\Rightarrow \mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$ satisfies Theorem.

$\Rightarrow \exists\, 0 \neq x \in L$, $\mathrm{ad}_L(x) = 0$

$\Rightarrow x \in Z(L) \neq 0$

By induction on $\dim L$

$L/Z(L)$ nilp.

$\Downarrow$

L nilp.

□

Def (Flag)

If $\dim V = n$, a flag in V is a series

$$0 \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

$$\dim V_i = i$$

Corollary.

$L \subseteq gl(V)$, $\forall x \in L$ nilp.

$\Rightarrow \exists$ flag in V, s.t. $x V_i \subseteq V_{i-1}$

Proof: $\exists\, 0 \neq v$, $L.v = 0$

$V_1 = Fv \qquad W = V/V_1$

$\varphi : V \to W$

$W_0 \subseteq W_1 \subseteq \cdots$

$V_i = \varphi^{-1}(W_{i-1})$

Lemma. L nilp. $K \triangleleft L$, If $K \neq 0$,

$\Rightarrow \quad K \cap Z(L) \neq 0$

Pf: L acts on K by

$$\varphi : L \to gl(K)$$

$$x \longrightarrow ad_x|_L$$

$\Rightarrow \exists\ 0 \neq x \in K \quad L.x = 0$

$\Leftrightarrow [L, x] = 0 \qquad x \in Z(L)$

Chap II Semisimple Lie Algebra.

§4. Theorem of Lie and Cartan.

§4.1. Lie's theorem.

Char $\mathbb{F} = 0$, $\mathbb{F} = \overline{\mathbb{F}}$

Theorem 4.1 Let L be a solvable subalg of $gl(V)$, if $V \neq 0$, then

$\exists\, 0 \neq v \in V$, $\lambda \in L^* = \mathrm{Hom}(L, \mathbb{F})$, s.t.

$x.v = \lambda(x)\, v$, $\forall x \in L$

(Common eigenvector)

Pf: By induction on $\dim L$

If $\dim L = 0$ ✓

$\dim L = 1$ ✓

Assume $\dim L \geq 2$

L is solvable

$\Rightarrow [L, L] \subsetneq L$

$[L/[L,L], L/[L,L]] = \bar{0}$

Suppose K is a subspace of L,

$[L, L] \subseteq K$

$[K, K] \subseteq [L, K] \subseteq [L, L] \subseteq K$

$\Rightarrow K \trianglelefteq L$

Take such K with $\dim \mathcal{L}/K = 1$,

K is solvable

By induction, $W = \{ v \in V \mid x.v \in \mathbb{F}v, \ \forall x \in K \} \neq 0$.

$\mathcal{L} = K \oplus \mathbb{F} z$, $\exists z \in \mathcal{L}$

Claim: $zW \subseteq W$

We need $\forall x \in K, v \in W$

$$x(zv) \in \mathbb{F}(zv)$$
$$\|$$
$$[x,z]v + zxv \in \mathbb{F}(zv)$$

Lemma. For any $x \in L$, $y \in K$, $w \in W$

We have $[x,y].w = 0$

Proof. $w, xw, x^2 w$

$W_0 = 0$

$W_1 = \text{Span}\{w\}$

$W_2 = \text{Span}\{w, xw\}$

Suppose $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_d \subseteq W_{d+1}$

$\{w, \cdots, x^{d-1} w\}$ is a basis of W_d

$yW_0 = yW_0$

$yW_1 = yu \subseteq W_1$

$$y \cdot xu = \underbrace{[y,x]}_{\overset{\uparrow}{K}} w + xyu \subseteq [y,x]u + xW_1$$

$$\subseteq \mathbb{F}u + \mathbb{F}xu \subseteq W_2$$

Claim: $yW_i \subseteq W_i$, $\quad [y,x](\underbrace{x^{i-1}w}_{\overset{\uparrow}{w_i}}) \in W_i$

$\Updownarrow$

Induction

$yx^{i-1}u \in W_i$

$$yx^iw = \underbrace{[y,x]}_{\overset{\uparrow}{K}} x^{i-1}w + x\boxed{y \cdot \underbrace{x^{i-1}w}_{w_i}}$$

$$\in W_i + W_{i+1} = W_{i+1}$$

Claim: $\forall y \in K,\ y(x^i w) - \lambda(y) x^i w \in W_i$

$(yw = \alpha(y) w,\ \forall y \in K)$

Induction. suppose

$$y(x^i w) - \lambda(y) x^i w \in W_i$$

$$y(x^{i+1} w) - \lambda(y) x^i w = [y, x] x^i w + x y x^i w - \lambda(y) x^{i+1} w$$

$$\in W_{i+1} + x(\underbrace{y x^i w - \lambda(y) x^i w}_{\in W_i})$$

$\in W_{i+1}$

$\{w, \dots, x^{d-1} w\}$ is a basis of

W_d

$$y(w \;\cdots\; x^{d-1} w) = (w, \dots, x^{d-1} w) \cdot \underset{\substack{\shortparallel \\ A\,.}}{\begin{pmatrix} \lambda(y) & & & * \\ 0 & \lambda(y) & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda(y) \end{pmatrix}}$$

$$[y,x] \sim \begin{pmatrix} \lambda([x,y]) & & * \\ & \ddots & \\ 0 & & \lambda([x,y]) \end{pmatrix} = B$$

$$xW_d \subseteq W_d$$

$$x \sim C$$

$$B = AC - CA$$

$$\Rightarrow 0 = tr(B) = d\,(\lambda([y,x]))$$

$$\Rightarrow [y,x]w = 0$$

$$L = K \oplus \mathbb{F}z$$

$\forall w \in W$, $x \in K$

$$x \cdot z \cdot w = ([x,z])(w) + zxw$$

$= zxw \in \overline{\mathbb{F}} zw$

$\Rightarrow \quad zw \subseteq W$

$\exists \; 0 \neq v \in W \quad \lambda_0 \in \mathbb{F}$

$zv = \lambda_0 v$ (This step use algebraic closeness).

Def: $u : L \to \overline{\mathbb{F}}$

$u(y) = \lambda(y)$

$u(z) = \lambda_0$

$\square$

$L \in gl(V)$ solvable

$0 \neq v$. $xv = \lambda(x) v$

$\forall x \in L$

$\{v, v_1, \cdots v_n\}$. $\begin{pmatrix} \lambda & * \\ & \boxed{*.} \end{pmatrix}$

$\exists$ Corollary. (Lie's theorem).

$L \in gl(V)$ solvable

$\Rightarrow \exists$ flag $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$

$\dim V_i = V$, s.t.

$xV_i \subseteq V_i$, $\forall x \in \mathcal{L}$, $i = 1 \sim n$.

(Trangulation in common).

Pf: Induction.

Quotien space.

□

Corollary 4.3.

$\mathcal{L}$ is Solvable

$\Rightarrow$ $\exists$ ideals of $\mathcal{L}$

$0 \subseteq L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k = L$

$\dim L_i = i$

Pf: $ad : L \to gl(L)$

$ad(L) < gl(L)$ solvable

By Cor 4.2, find L_i

$ad_L(L_i) \subseteq L_i \iff [L, L_i] \subseteq L_i$

Corollary 4.4.

L solvable, then $\forall x \in [L, L]$,

$ad x$ is nilpotent.

Proof. L solvable $\Rightarrow \mathrm{ad}\, L \subseteq gl(L)$

$\Rightarrow \exists\, g \ , \ g\, \mathrm{ad}\, L\, g^{-1} \subseteq t_n(\mathbb{F})$

$\Rightarrow \ g\, [\mathrm{ad}\, L, \mathrm{ad}\, L]\, g^{-1}$

$= [g\, \mathrm{ad}\, L\, g^{-1}, \ g\, \mathrm{ad}\, L\, g^{-1}] \subseteq n_n(\mathbb{F})$

$\forall\, x \in [L, L]$

$g\, \mathrm{ad}_x\, g^{-1} \in n_n(\mathbb{F})$ is nilpotent.

§4.2. Jordan-Chevalley decomposition

$x \in gl(V) = \mathrm{End}(V)$ $\exists$ Basis.

$x \sim A = \mathrm{diag}(J_{m_1}(\lambda_1), \dots, J_{m_k}(\lambda_k))$

($\mathbb{F}$ is required to have char $\neq 0$ and algebraically closed).

$J_m(\lambda) = \lambda I_m + N_m$

$= X + Y$

$XY = YX$, $Y^m = 0$

Define 4.5.

$x \in \text{End}\, V$ is semisimple

$\Leftrightarrow (d_x(\lambda), d_x'(\lambda)) = 1$

(Minimal polynomial has no multiple roots).

Remark. $\mathbb{F} = \overline{\mathbb{F}}$

(1) x semisimple $\Leftrightarrow$ x is diagonalizable

(2) x, y is semisimple, $xy = yx$

$\Rightarrow$ $x \pm y$ semisimple

(3) x semi. on V $W \subseteq V$

$xW \subseteq W$

$\Rightarrow x|_W$ is semi.

Prop. 4.7. $x \in \mathrm{End}(V)$.

(a) There are unique

$x_n, x_s \in \mathrm{End}(V)$, x_n nilpotent,

x_s semi. $x_n x_s = x_s x_n$

(b) $\exists\, p(t), q(t) \in \mathbb{F}[t]$

$p(0) = q(0) = 0$, s.t.

$x_s = p(x)$, $x_n = q(x)$, $x_n + x_s = x$

(c) If $A \subseteq B \subseteq V$, $x(B) \subseteq A$

$\Rightarrow x_s(B) \subseteq A$, $x_n(B) \subseteq A$

$(x^k(B) \subseteq A, \forall k \geq 1)$

If (b) holds:

$x = x_s + x_n = p(x) + q(x)$

If $x_s - x_s' = x_n' - x_n$

$x_s' x_n' = x_n' x_s'$, x_s' semi.

x_n' nil.

$\Rightarrow$ $x_s' x = x x_s'$

$x_n' x = x x_n'$

$\Rightarrow$ $x_s' x_s = x_s x_s'$ ……

$x_s' x_n = x_n x_s'$

$\Rightarrow$ $x_s - x_s' = x_n' - x_n$

is both semi. and nil.

$\Rightarrow$ $x_s = x_s'$, $x_n' = x_n$.

Proof. (b) $\overline{\mathbb{F}} = \mathbb{F}$

$x \in \mathrm{End}(V)$, $\varphi_x(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$

$\lambda_i \neq \lambda_j$.

$\varphi_i := (x-\lambda_i)^{m_i}$, $V_i = \ker(t-\lambda_i)^{m_i}$

$V = V_1 \oplus \cdots \oplus V_k$

By Chinese Remainder theorem,

$\exists\, p(t) \in \mathbb{F}[t]$, s.t.

$p(t) \equiv \lambda_i \mod \varphi_i(t)$

$p(t) \equiv 0 \mod t$.

$(t \mid \varphi_i(t) \Leftrightarrow \lambda_i = 0)$ ✓.

$q(t) = t - p(t)$

$x_s \triangleq p(x) \qquad x_n \triangleq q(x)$

$x_s|_{V_i} = \lambda_i \, Id \Rightarrow x_s$ is semisimple.

$x_n|_{V_i} = (x - x_s)|_{V_i} = (x - \lambda_i)|_{V_i}$

$\Rightarrow x_n$ is nilpotent.

☆ $ad: L \to gl(L)$

$ad_x = (ad\,x)_s + (ad\,x)_n$

If $Z(\mathcal{L})=0$

$\Rightarrow$ ad 1 to 1

If $[\mathcal{L},\mathcal{L}]$ nilpotent

$\Rightarrow \mathcal{L}$ solvable

---

x_s is called the s.s. part of

x

x_n is called the nil. part of

Example. $x = x_s + x_n$, $x_s, x_n \in gl(V)$

$ad_x = ad_{x_s} + ad_{x_n} \in End(End(V))$

x_n nil. ⇒ $ad\, x_n$ nil.

x_s s.s. ⇒ $ad\, x_s$ s.s.

$\exists (v_1 \sim v_n)$ basis of V,

$x_s(v_1 \cdots v_n) = (v_1 \cdots v_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$e_{ij}(v_k) = \delta_{jk} v_i$

⇒ $ad\, x_s(e_{ij}) = (\lambda_i - \lambda_j) e_{ij}$

⇒ ad_{x_s} s.s.

$[ad\, x_s, ad\, x_n] = ad_{[x_s, x_n]} = 0$

$\Rightarrow$ $\mathrm{ad}\,x = \mathrm{ad}\,x_s + \mathrm{ad}\,x_n$

is Jordan - chevalley decomposition.

$\Rightarrow (\mathrm{ad}\,x)_s = \mathrm{ad}\,x_s$, $x \in gl(V)$

Lemma 4.9. $x \in gl(V)$

$x = x_s + x_n \Rightarrow \mathrm{ad}\,x = \mathrm{ad}\,x_s + \mathrm{ad}\,x_n$

Lemma. 4.10.

A is an F-alg , $\forall \sigma \in \mathrm{Der}(A)$

$\sigma = \sigma_s + \sigma_n$ is the Jordan decom.

in $gl(A)$

$\Rightarrow \sigma_s, \sigma_n \in \mathrm{Der}(A)$

Pf: $A = \bigoplus_{a \in \mathbb{F}} A_a \qquad A_a = \ker(\sigma - a)^n$

Claim: $A_a, A_b \neq 0$

$\Rightarrow A_a A_b \subseteq A_{a+b}$

$$\sigma_s\Big|_{A_a} = a.\mathrm{Id}\Big|_{A_a}$$

$x \in A_a, \; y \in A_b$

$(\sigma - a)^k x = 0 \qquad (\sigma - b)^l y = 0$

$\Rightarrow (\sigma - (a+b))^{l+k} (xy)$

$$= \sum_{i=0}^{l+k} C_{l+k}^{i} \left((\sigma - a)^{l+k-i}(x)\right)\left((\sigma-b)^{i}(y)\right)$$

$\nearrow$

(Induction.).

$\square$

$$\sigma_s(xy) = (a+b)xy = axy + xby$$

$$= (\sigma_s x)y + x(\sigma_s(y))$$

$\Rightarrow \sigma_s \in Der(A)$

$\Rightarrow \sigma_n \in Der(A)$

§4.3. Cartan's Criterian.

Lemma. 4.11

$A \subseteq B \subseteq gl(V)$

$$M \overset{\Delta}{=} \{ x \in gl(V) \mid [x, B] \subseteq A \}$$

Suppose $x \in M$, $\forall y \in M$

$tr(xy) = 0$, then x is nilpotent.

pf: $x \in gl(V)$, $x = x_s + x_n$

x nil $\Leftrightarrow$ $x_s = 0$

$$x_s = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Assume $(\lambda_1, \cdots, \lambda_n) \neq (0, \cdots, 0)$

$E = \mathrm{Span}_{\mathbb{Q}} \{ \lambda_1, \cdots, \lambda_n \}$

($\mathbb{F}$ is required to have char $\mathbb{F} \neq 0$

then $\mathbb{Q} \hookrightarrow \mathbb{F}$.)

$E = 0$

$\Leftrightarrow \quad \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{Q}) = 0$

$\forall f \in E^*$

take $y \in \mathfrak{gl}(V)$

$$y \longrightarrow \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}$$

$$x_s \longrightarrow \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\mathrm{ad}_{x_s}(e_{ij}) = (\lambda_i - \lambda_j)\, e_{ij}$$

$$\mathrm{ad}_y(e_{ij}) = (f(\lambda_i) - f(\lambda_j))\, e_{ij}$$

$$= f(\lambda_i - \lambda_j) e_{ij}$$

$$\begin{array}{ccc} \lambda_1 - \lambda_2 & & \lambda_1 - \lambda_k \\ & \cdots & \\ f(\lambda_1 - \lambda_2) & & f(\lambda_1 - \lambda_k) \end{array}$$

$\Rightarrow \exists\, r(t) \in \mathbb{F}[t]$, s.t.

- $r(0) = 0$
- $r(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j)$, i, j.

$$\operatorname{ad} y \; e_{ij} = r(\lambda_i - \lambda_j) e_{ij}$$

$$\operatorname{ad} x_s \; e_{ij} = (\lambda_i - \lambda_j) e_{ij}$$

$$\Rightarrow \operatorname{ad} y = r(\operatorname{ad} x_s)$$

$\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ Jordan Decomposition

of $\mathrm{ad}\, x$ in $\mathrm{End}(\mathrm{End}(V))$

$\mathrm{ad}\, x_s = p(\mathrm{ad}\, x)$ $p(0) = 0$, $p \in \mathbb{F}[t]$

$\Rightarrow \mathrm{ad}\, y = r(p(\mathrm{ad}\, x)) = \tilde{p}(0) = 0$

$\mathrm{ad}\, x(B) \subseteq A$

$\Rightarrow \mathrm{ad}\, y(B) \subseteq A$

$\Rightarrow y \in M$

$\Rightarrow \mathrm{tr}(xy) = 0$

$\Rightarrow \sum_{i=1}^{n} f(\lambda_i)\lambda_i = 0$

$$\Rightarrow \quad 0 = f\left(\sum_{i=1}^{n} f(\lambda_i)\lambda_i\right)$$

$$= \sum_{i=1}^{n} f(\lambda_i) f(\lambda_i)$$

$$\Rightarrow \quad f(\lambda_i) = 0, \ \forall i$$

$$\Rightarrow \quad f = 0 \Rightarrow E^* = 0 \Rightarrow E = 0 \Rightarrow x_s = 0.$$

$$\square$$

Remark. What the fuck is this proof?

Theorem. 4.11 (Cartan's

criterian)

$L \subseteq gl(V)$, Suppose $tr(xy)=0$,

$\forall x \in [L,L],\ y \in L \Rightarrow L$ solvable

Pf: $A=[L,L]$ $B=L$

$M=\{x \in gl(V) \mid [x,L] \subseteq [L,L]\} \supseteq L$

Claim: $\forall x \in [L,L],\ y \in M,\ tr(xy)=0$

$x=\sum_i [a_i,b_i]$ $a_i, b_i \in L$, $y \in M$

$tr(xy) = \sum_i tr([a_i,b_i]y)$

$$\operatorname{tr}([A,B]C) = \operatorname{tr}(ABC - BAC)$$

$$= \operatorname{tr}(ABC - ACB)$$

$$= \operatorname{tr}(A[B,C])$$

$$\Downarrow$$

$$= \sum_i \operatorname{tr}(\underset{\overset{\uparrow}{\mathcal{L}}}{a_i}\,\underset{\overset{\uparrow}{[\mathcal{L},\mathcal{L}]}}{[b_i, y]}) = 0$$

By Lemma 4.11

$x \in \mathfrak{gl}(V)$ nil.

$\Rightarrow$ x is ad-nil.

$\Rightarrow$ $[\mathcal{L},\mathcal{L}]$ nil. (Engel's thm

$\Rightarrow$ $\mathcal{L}$ solvable

Corollary 4.13.

L finite dim

If $\operatorname{tr} ad_x ad_y = 0$

$\forall x \in [L, L], \ y \in L$

$\Rightarrow L$ solvable

Pf: $ad\, L < gl(L)$

$\operatorname{tr} ad_x ad_y = 0, \ \forall ad_x \in [ad\, L, ad\, L]$

$ad_y \in ad\, L$

Cartan Criterion

$\Rightarrow$ ad $\mathcal{L}$ solvable.

§5. Killing form.

$0 \neq \mathcal{L}$ finite dim, char $\mathbb{F} = 0$

The following are equivelant:

(1) $\mathcal{L}$ semisimple

(2) $\mathcal{L}$ has no abelian ideal

(3) The $\underline{\text{killing form}}$ is non-degenerated
$K: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$

(4) $\mathcal{L}$ is direct sum of simple ideal.

Def. 5.1. Killing form.

$K(x,y) = \mathrm{tr}\ \mathrm{ad}_x \mathrm{ad}_y$

$\Rightarrow$ (1) $K(x,y) = K(y,x)$

(2) Bilinear

(3) $K([x,y],z) = K(x,[y,z])$

"Associativity"

Prop 5.2.

$\varphi \in \mathrm{Aut}(L) \subseteq \mathrm{End}(L)$

$\Rightarrow K(\varphi(x), \varphi(y)) = K(x,y)$

Pf: $K(\varphi(x), \varphi(y))$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathrm{ad}\,\varphi(x)$

$= \mathrm{tr}\, \mathrm{ad}_{\varphi(x)} \mathrm{ad}_{\varphi(y)} \quad = \varphi(\mathrm{ad} x)\varphi^{-1}$

$= \mathrm{tr}(\varphi\, \mathrm{ad}_x \mathrm{ad}_y \varphi^{-1})$

$= K(x, y)$

Fact: $W \subseteq V \quad \varphi \in \mathrm{End}(V)$

$\varphi(V) \subseteq W$

$\Rightarrow \mathrm{tr}\,\varphi = \mathrm{tr}\,\varphi|_W$

Lemma. 5.3.

$I \trianglelefteq L$

$K_I : I \times I \to \mathbb{F}$

$K_I(x, y) = tr(ad_I x \, ad_I y)$

$\Rightarrow K = K_I$

Pf: $I \subseteq L$

$ad_x, ad_y : L \to I$

$tr(ad_x \, ad_y) = tr(ad_x ad_y)|_I$

$= tr(ad x|_I \, ad y|_I)$

$= K_I(x, y)$

Define 5.4. Non-degenerate.

$\beta : L \times L \to \mathbb{F}$ is called ~

If $\{x \in L \mid \beta(x,y)=0, \forall y \in L\} = 0$.

S_β " radical

Prop. 5.5.

$S_K \triangleleft L$ $\quad S_K = \{x \in L \mid K(x,y)=0, \forall y\}$

$\forall x \in S_K, \ z \in L$

$\forall y \in L$

$K([x,z], y) = K(x, [z,y]) = 0$

$\Rightarrow [x,z] \in S_K$

Lemma. 5.6.

$I \triangleleft L, \quad [I,I] = 0$

$\Rightarrow I \subseteq S_K$

$((3) \Rightarrow (2))$

pf: $\forall x \in I, \quad y \in L$

$K(x, y) = tr(ad_x ad_y)$

$ad_x ad_y : L \to L$

$(ad_x ad_y)^2 (L)$

$\subseteq ad_x ad_y (I) \subseteq ad_x (I) = 0$

$\Rightarrow K(x, y) = 0$

Lemma 5.7.

L semisimple $\Leftrightarrow$ L has no abelian ideal

Pf: $\Rightarrow$: ✓

$\Leftarrow$: $\text{Rad}\, L = J$, J solvable

If $J \neq 0$

$\Rightarrow \exists\, K$, $J^{(k)} \neq 0$, $J^{(k+1)} = 0$

$J^{(k+1)} = [J^{(k)}, J^{(k)}] = 0$

$\Rightarrow J^{(k)}$ is abelian.

theorem 5.9.

$\mathcal{L}$ is s.s. $\Leftrightarrow$ $S_K = 0$

pf: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) : ✓

(1) $\Rightarrow$ (3) :

Claim: $S_K \subseteq Rad(\mathcal{L})$

(1) $S_K \triangleleft \mathcal{L}$

(2) $K(x,y)=0, \ \forall x \in S_K, \ \forall y \in \mathcal{L}$

$\Rightarrow K(x,y)=0, \ \forall x \in S_K, \ \forall y \in [S_K, S_K]$

$\Big\Vert$ Cor 4.13.

$$\Downarrow$$

S_k solvable

$\square$

§5.2. Simple ideals of L

$L = L_1 \oplus \cdots \oplus L_k$

L_i simple $L_i \trianglelefteq L$

If $I \trianglelefteq L$, I simple

$\Rightarrow \exists i,\ I = L_i$

Pf: $I \trianglelefteq L$ simple

$\Rightarrow [I, I] \neq 0 \quad \sum [I, I_i]$

$\Rightarrow \exists i, \ [I, I_i] \neq 0$

$\Rightarrow [I, I_i] = I_i = I$

Lemma. $L \neq 0$ $\boxed{\text{s.s.}}$ Lie alg.

$I \triangleleft L \Rightarrow \exists \ I^{\perp} \triangleleft L$, s.t.

$L = I \oplus I^{\perp}$

Specially, if $J \triangleleft I$, then $J \triangleleft L$

proof: L s.s.

$\Rightarrow S_K = 0$

$I^{\perp} = \{x \in L \mid K(x,y) = 0, \forall y \in I\}$

Claim: $I^{\perp} \triangleleft L$, $I \cap I^{\perp} = 0$

$\forall x \in I^{\perp}, \forall z \in L, \forall y \in I$

$K([x,z],y]) = K(x,[z,y]) = 0$

$\forall x, y \in I \cap I^{-}$

$0 = K(x,y) = tr(\operatorname{ad}x \operatorname{ad}y)$

$\Rightarrow I \cap I^{\perp} \triangleleft L$ solvable

$\square$

Theorem 5.13. $L \neq 0$ s.s.

Then $\exists L_1, \cdots, L_k \trianglelefteq L$, L_i simple

$$L = L_1 \oplus \cdots \oplus L_k$$

Moreover. if $I \trianglelefteq L$ simple

$\Rightarrow \exists i,\ I = L_i$

Pf: Induction on $\dim L$

If L simple, ✓

Otherwise, take a minimal non-zero ideal I of L, then $I \neq L$

$\Rightarrow \mathcal{L} = I \oplus I^{\perp}$

By Lemma 5.11.

$\forall J \triangleleft I, \Rightarrow J \triangleleft \mathcal{L} \quad I, I^{\perp}$ s.s.

$I = ? \oplus \cdots \oplus ?$

$I^{\perp} = ? \oplus ? \oplus \cdots \oplus ?$

Each ? is simple.

* * *

If $I \triangleleft \mathcal{L}$ simple

$\Rightarrow [I, \mathcal{L}] \supseteq [I, I] \neq 0.$

$\Rightarrow \exists [I, L_i] \neq 0$

$\Rightarrow [I, L_i] \triangleleft I, L_i$

$\Rightarrow I = L_i$

Moreover $\kappa|_{L_i} = (\kappa_L)|_{L_i \times L_i}$

Cor. 5.14.

If L is s.s. then $L = [L, L]$ and all ideals and hom. image are s.s.

Moreover, each ideal is a

direct sum of certain simple ideals

$[L,L] \supseteq \sum [L_i, L_i] = L.$

Lemma 5.11

$\Rightarrow I, I^{\perp}$ semisimple

$I = I_1 \oplus \cdots \oplus I_?$

$\Rightarrow I_i = L_j. \quad \exists j$

$L/I = L_? \oplus \cdots \oplus L_?$

Cor 5.15. (1) $\Leftrightarrow$ (4)

(1) $\Rightarrow$ (4): ✓

(4) $\Rightarrow$ (3):

$L = L_1 \oplus \cdots \oplus L_k \quad L_i \trianglelefteq L$ simple

✓.

§ 5.3. inner derivations (ad_x).

$\mathrm{ad}\, L \trianglelefteq \mathrm{Der}(L)$

If L semisimple

$\mathrm{ad} : L \to \mathfrak{gl}(L)$ is injective.

$\Rightarrow L \cong \mathrm{ad}\, L \trianglelefteq \mathrm{Der}(L)$

Theorem 5.16

If L s.s.

$\Rightarrow \operatorname{ad} L = \operatorname{Der}(L)$

Pf: Let $M = \operatorname{ad} L \triangleleft \operatorname{Der}(L) = D$

$M \cong L$ is s.s.

$\Rightarrow \kappa_M$ non deg.

$\kappa_D|_{M\times M} = \kappa_M$ (Because $M \triangleleft D$).

Define $M^\perp = \{x \in D \mid \kappa_D(x,y) = 0, \forall y \in M\}$

$\Rightarrow M^{\perp} \trianglelefteq D$, $M \cap M^{\perp}$ is solvable

ideal of M

$\Rightarrow M \cap M^{\perp} = 0$.

$\dim M^{\perp} \geq \dim D - \dim M$

$\Rightarrow D = M \oplus M^{\perp}$

$\forall \delta \in M^{\perp}$ $\quad$ $\operatorname{ad} x \in M = \operatorname{ad} L$

$[\delta, \operatorname{ad} x] = \operatorname{ad}_{\delta(x)} \in M \cap M^{\perp}$

$\Rightarrow \operatorname{ad}_{\delta(x)} = 0$

$\Rightarrow \delta(x) = 0$ $\quad$, $\forall x$

$\Rightarrow \delta = 0.$

§6.4. Abstract Jordan decomposition

If L is semisimple

$\Rightarrow \operatorname{ad} L = \operatorname{Der}(L)$

$\delta \in \operatorname{Der} L \qquad \delta = \delta_s + \delta_n \in \mathfrak{gl}(L)$

$\Rightarrow \delta_s, \delta_n \in \operatorname{Der}(L)$ (Lemma 4.10)

$(\operatorname{ad} x)_s = \operatorname{ad} \underset{\sim}{x_s}$

$(\operatorname{ad} x)_n = \operatorname{ad} \underset{\sim}{x_n}$

$0 = [(\mathrm{ad}\, x)_s, (\mathrm{ad}\, x)_n]$

$= \mathrm{ad}\, [x_s, x_n]$

$\Rightarrow [x_s, x_n] = 0$

$\exists!\ x_s, x_n$, s.t.

x_s ad - s.s.

x_n ad - nil. $[x_s, x_n] = 0$

$x = x_s + x_n$

Abstract Jordan Decomposition.

Example 5.18.

$0 \neq I \trianglelefteq \mathfrak{sl}_n(\mathbb{F})$

$\exists\ 0 \neq A = (a_{ij}) \in I \quad a_{rs} \neq 0$

$h = \mathrm{diag}(1, \cdots, 2^{n-2}, 1-2^{n-1})$

$[h, e_{ij}] = (2^{i-1} - 2^{j-1}) e_{ij}.$

$\Rightarrow e_{ij} \in I$

$\forall\ x \in sl(V) \quad x = x_s + x_n$ abstract J-D.

$x \in sl(V) \subseteq gl(V)$

$x = s + n$ in $gl(V)$

$\Rightarrow \mathrm{tr}\, n = 0,\ \mathrm{tr}\, s = 0$

$\Rightarrow$ $s, n \in sl(V)$

$\Rightarrow$ $ad\,s = ad\,x_s$, $ad\,n = ad\,x_n$

$\Rightarrow$ $s = x_s$, $n = n_n$

$\Rightarrow$ "Abstract" J-D as same as real J-D.

---

§6. Complete reducibility of representation.

$\mathcal{L}$ is s.s.

$\varphi : \mathcal{L} \to gl(V)$

$\Rightarrow V = \oplus V_i$, V_i irre. repr.

§6.1 Modules.

Define. 6.1. A vector space V with an operation

$$\mathcal{L} \times V \to V$$

$(x, v) \to x.v$ is called $\mathcal{L}$-module

(M1) $(ax+by).v = a\, x.v + b\, y.v$

(M2) $x.(av+bu) = a x.v + b x.u$

(M3) $[x,y].v = x.(y.v) - y.(x.v)$

Remark. 6.2.

(1) If $\varphi: L \to gl(V)$ is a hom.

(repr.)

then $x.v := \varphi(x)(v)$

gives an L-module structure

Conversely, If V is an L-module

$\varphi(x)(v) := x.v$, φ is a repn.

(2) V is an L module.

Define $g = L \oplus V$ as vector spaces

$\forall\ x, y \in L,\ u, v \in V$

$$[(x,u),(y,v)] := ([x,y], xv - yu) \in L \oplus V$$

Then g is a Lie alg.

This is called the semi-direct product.

$$g = L \ltimes V$$

$G = G_1 \ltimes G_2 \Leftrightarrow G_1 G_2 = G$, $G_2 \triangleleft G$

$G_1 \cap G_2 = \{1\}$

(3) In general, $\mathcal{L}, K$ Lie algs

$\varphi : \mathcal{L} \to \mathrm{Der}(K)$ is Lie-alg hom.

$g = \mathcal{L} \ltimes K = \mathcal{L} \oplus K$ (as vector space).

Define

$$[(x,a),(y,b)] = ([x,y], [a,b] + \varphi(x)(b) - \varphi(y)(a))$$

is a Lie-alg

(*) $K = V$ is an $\mathcal{L}$-module.

Let $ab=0$, $\forall a,b \in V \Rightarrow \mathrm{Der}(K)=\mathfrak{gl}(K)$.

$\leftarrow$ associated alg.

then (2) E B/.

Define 2.3.

(1) V, W are L-modules

A hom. of L-mod is linear map

$\psi: V \to W$

$$\psi(x.V)=\psi(x).V$$

(2) An iso. of modules is a bijective hom.

If $\exists$ iso. $\varphi : V \to W$

Then we call V and W are equivelent

$\mathcal{L}$-modules

(3). Submodules.

V is an $\mathcal{L}$-module. W is a subspace of V, if $\forall x \in \mathcal{L}$ $xW \subseteq W$

W is called a submodule of V

(Invariant subspace).

V is $\mathcal{L}$-module

$\Leftrightarrow \varphi: \mathcal{L} \to \mathfrak{gl}(V)$ is a hom.

W is a submodule

$\Leftrightarrow \varphi$ induces $\varphi_W: \mathcal{L} \to \mathfrak{gl}(W)$

(i.e. W is an $\mathcal{L}$-invariant subspace).

$$\forall x \in \mathcal{L},\quad \varphi(x)(\overbrace{v_1}^{w} \overbrace{\cdots v_n}^{v.}) = (v_1 \cdots v_n)\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$\Rightarrow \varphi(\mathcal{L}) \subseteq \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

If $\mathcal{L}$-mod V has precisely 2

$\mathcal{L}$-submod $(V, 0)$, then V is called an irre. $\mathcal{L}$-module.

(4). V, W are $\mathcal{L}$-mod $U = V \oplus W$

$x.(v,w) = (x.v, x.w)$

$\Rightarrow U$ is an $\mathcal{L}$-mod.

(5) Completely reducible.

An $\mathcal{L}$-mod V is called completely reducible, if $V = V_1 \oplus \cdots \oplus V_k$, s.t.

V_i is irreducible.

(Exercise 25).

V is completely reducible

$\Leftrightarrow \forall$ L-submodule W of V, $\exists$

L-submodule W_x, s.t. $V = W \oplus W_x$

Remark 6.4. (1)

These notions are all standard and also make sense when $\dim V = +\infty$

(2) $\varphi: L \to gl(V)$

$\varphi(x) \in gl(V) = \mathrm{End}(V)$

Define $A_{L,\varphi} = < \varphi(x) \mid x \in L >$ is a subalg.

of $\mathrm{End}(V)$

V is an irre. L-module

$\Leftrightarrow$ V is an irre. $A_{L,\varphi}$-module

(3) If $\varphi: V \to W$ is an L-mod hom.

then $\ker\varphi$ is a submod of V

$\mathrm{im}\,\varphi$ is a submod of W -

(4) Jordan-Holder thm holds for L.

Theorem 6.5 (Schur's Lemma).

Let $\varphi: L \to gl(V)$ be irre.

then the only endomorphism of

V commuting with all $\varphi(x)$ are

scalars. $(\mathbb{F} = \overline{\mathbb{F}})$

Pf: Let $f \in End(V)$ such that

$[f, \varphi(x)] = 0$ for $\forall x \in L$

$f \in End(V)$

$\Rightarrow \exists \lambda \in \overline{F}$, $0 \neq v \in V$ s.t.

$f(v) = \lambda V$

Set $g = f - \lambda \mathrm{Id}$

$\Rightarrow \ker g \neq 0$.

$\forall x \in \mathcal{L}$, $w \in \ker g$

$$(f - \lambda \mathrm{Id})\, x w = x (f - \lambda \mathrm{Id})\, w = 0$$

$\Rightarrow \ker g$ is $\mathcal{L}$-invariant.

$\Rightarrow \ker g = V$

$\square$

Example 6.6.

L is an L-module

L-submods of L $\leftrightarrow$ ideals

If L is simple $\Rightarrow$ L is an irre.

L-module

Example

(1) If V is an L-module

$V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$

define : $\forall f \in V^*$, $x \in L$

$$(x.f)(v) = -f(x,v)$$

$$([x,y].f)(v) = -f([x,y]v)$$

$$= -f(xyv - yxv)$$

$$= x.y.f(v) - y.x.f(v)$$

(3) V, W are L-module

$$U \triangleq V \otimes W$$

$$x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w).$$

(Compare to grp rep^n.)

$$\left(\quad g(v\otimes w) = g(v) \otimes g(w) \quad \right).$$

Check: $(M_1)(M_2)$ ✓.

$$[x,y](v\otimes w)$$

$$= [x,y]v \otimes w + v \otimes [x,y]w$$

$$= (x.y.v - y.x.v) \otimes w + v \otimes (x.y.w - y.x.w)$$

$$= x.y.(v\otimes w) - y.x.(v\otimes w)$$

(3) V, W are L-modules.

$$\mathrm{Hom}_F(V,W) \xrightarrow{\sim} V^* \otimes W$$

Define $x\in L$, $f\in \mathrm{Hom}_F(V,W)$

$$(x.\varphi)(v) = x.\varphi(v) - \varphi(x,v)$$

Remark: The annihilator of this space is $\mathrm{Hom}_L(V,W)$!

Claim: $\mathrm{Hom}_F(V,W)$ is an L-module.

$(M_1)(M_2)$ ✓.

(M_3):

$$([x,y].\varphi)(v)$$

$$= x.y.\varphi(v) - y.x.\varphi(v)$$

$$-\varphi(x.y.v - y.x.v)$$

$$= x.y.\varphi(v) - \varphi(x.y.v)$$

$$-(y.x.\varphi(v)) - \varphi(y.x.v).$$

$$\mathrm{Hom}_F(V,W) \xrightarrow{\sim} V^* \otimes W$$

$$(\sigma : z \mapsto f(z)w) \xleftarrow{\varphi} f \otimes w$$

$\dim V, W < +\infty.$

is an isomorphism.

$$\varphi(x.(f\otimes w))(v) = \varphi((x.f)\otimes w + f\otimes(x.w))(v)$$

$$= (x.f)(v)w + f(v)(x.w)$$

$$= -f(x.v)w + f(v)(x.w)$$

$$= (x.\varphi(f\otimes w))(v).$$

$$(x.\varphi(f\otimes w))(v)$$

$$= x(\varphi(f\otimes w)(v))$$

$$-\varphi(f\otimes w)(xv)$$

§ 6.2 Casimir element of a repn.

Define 6.8. A rep. $\varphi: L \to gl(V)$ is called faithful

$\Leftrightarrow$ φ is injective

Example 6.9.

Let $\varphi: L \to gl(V)$ is faithful rep. of a s.s. Lie alg.

Define a symmetric bilinear form.

$\beta(x,y) = \operatorname{tr}(\varphi(x)\varphi(y))$.

$\Rightarrow$ β is associative (or invariant)

i.e. $\beta([x,y],z) = \beta(x,[y,z])$

$\Rightarrow$ $\operatorname{Rad}(\beta)$ is an ideal. Moreover, it is solvable.

L is s.s. $\Rightarrow$ β is non-degenerate.

Let L be a s.s. Lie alg

$\beta: L \times L \to F$

is a non deg sym bilinear asso.

form

If $x_1 \sim x_n$ is a basis of L, then there is a Dual basis $y_1 \sim y_n$

$$\beta(x_i, y_j) = \delta_{ij}.$$

$\exists P$, $\beta(x,y) = u^{T}Py$

If $x \in L$, then

$$[x, x_i] = \sum_j \overset{A}{\boxed{a_{ij}}} x_j$$

$$[x, y_j] = \sum_j \boxed{b_{ij}}\, \underset{B.}{y_j}$$

$$\begin{array}{c} \beta([x,x_i], y_k) = a_{ik} \\ \shortparallel \\ -\beta(x_i, [x, y_k]) = -b_{ki} \end{array}$$

$\Rightarrow A^T = -B$

Assume $\varphi : L \to gl(V)$ is a faithful rep. of s.s. Lie alg.

$\{x_1, \cdots, x_n\}$ $\{y_1, \cdots, y_n\}$ are dual basis relative β.

Define $C_\varphi(\beta) = \sum_{i=1}^{n} \varphi(x_i)\varphi(y_i) \in End(V)$

Claim : (1) $C_\varphi(\beta)$ is independent of the choice of $\{x_i\}_i$

(2) For $\forall x \in L$,

$$[\varphi(x), c_\varphi(\beta)] = 0.$$

(Recall schur's lemma).

Pf of claim:

(1) $\{z_1 \sim z_n\}$ $\{w_1 \sim w_n\}$ $\beta(z_i, w_j) = \delta_{ij}$.

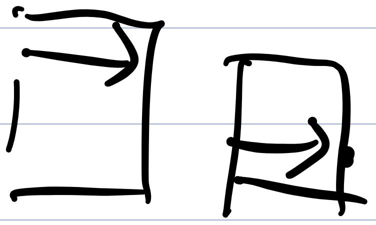
Assume $z_i = \sum_j c_{ij} x_i$

$$w_i = \sum_j d_{ij} y_j$$

$$\Rightarrow \delta_{lk} = \beta\left(\sum_i c_{ki} x_i, \sum_j d_{lj} y_j\right)$$

$$= \sum \delta_{ij} c_{ki} d_{lj}$$

$$= \sum_{i} c_{ki} d_{li}$$



$$\Leftrightarrow CD^{T} = I_n ,$$

$$\Leftrightarrow \sum_{k} d_{ki} c_{kj} = \delta_{ij}$$

$$\sum_{k} \varphi(z_k) \varphi(w_k)$$

$$= \sum_{i,j,k} c_{ki} d_{kj} \varphi(x_i) \varphi(y_j)$$

$$= \sum_{i,j} \sum_{k} c_{ki} d_{kj} \varphi(x_i) \varphi(y_j)$$

$$= \sum_{i,j} \delta_{ij} \varphi(x_i) \varphi(y_j)$$

$$= \sum_i \varphi(x_i)\varphi(y_i)$$

(2) In $gl(V)$

$$[x, yz] = [x,y]z + y[x,z]$$

(because ad_x is a derivation).

$$[\varphi(x), C_\varphi(B)] = \sum_{i=1}^{n} [\varphi(x), \varphi(x_i)\varphi(y_i)]$$

$$= \sum_{i=1}^{n} [\underline{\varphi(x), \varphi(x_i)}]\varphi(y_i) + \sum_{i=1}^{n} \varphi(x_i)[\varphi(\underline{x), \varphi(y_i)}]$$

$$= \sum_{i,j=1}^{n} \underline{a_{ij}\varphi(x_j)}\varphi(y_i)$$

$$+\sum_{i,k=1}^{n} b_{ik}\,\varphi(x_i)\varphi(y_k)$$

$$=0.$$

Define 6.10

$C_\varphi(\beta)$ is called Casimir element of φ.

Remark. 6.1.

(1) $\operatorname{tr} C_\varphi(\beta) = \sum_{i=1}^{n} \operatorname{tr} \varphi(x_i)\varphi(y_i)$

$$= \sum_{i=1}^{n} \beta(x_i, y_i) = n.$$

$$= \dim L$$

$\Rightarrow$ if φ is irre.

Apply Schur's Lemma,

$$C_\varphi(\beta) = \frac{\dim L}{\dim V} \cdot \mathrm{Id}$$

Example. 6.12. $L = sl_2(F)$

$V = F^2$

$\varphi : L \hookrightarrow gl(V)$

x, h, y basis of L

$\beta(A, B) \stackrel{\Delta}{=} \operatorname{tr} AB$

$\Rightarrow \{x, h, y\}, \{y, \frac{h}{2}, x\}$ is dual.

$$C_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{3}{2} \mathrm{Id} = \frac{\dim L}{\dim V} \mathrm{Id}.$$

Recall. Casimir element.

①. well-defined.

②. $[\varphi(x), C_\varphi(\beta)] = 0$

$\forall x \in L$

Remark. 6.B.

(1) If φ is inj. L is s.s.

$\Rightarrow$ $\ker \varphi \triangleleft L$ s.s.

$L = L_1 \oplus \dots \oplus L_k \oplus \dots \oplus L_t$ L_i simple

$\ker \varphi = L_{k+1} \oplus \dots \oplus_t$

Define $L' = L_1 \oplus \dots \oplus L_k$

$\varphi|_{L'}$ is faithful.

$C_\varphi(\beta)$ relative to $\varphi|_{L'}$

If V is an irr. L-module

$\Rightarrow$ V is an irr. L'-module

$$C_\varphi(\beta) = \frac{\dim L'}{\dim V} \, Id.$$

2) If L is simple.

$\varphi : L \to gl(V)$.

$\ker \varphi = 0$ or L

$\Downarrow$

$L.V \equiv 0$.

§6.3. Weyl's thm.

Lemma. 6.14. $\varphi : L \to gl(V)$ is a finite dim. rep. of s.s. L.

then $\varphi(L) \subseteq sl(V)$.

Pf: $\varphi(L) = [\varphi(L), \varphi(L)] \subseteq sl(V)$

Theorem 6.15.

Let $\varphi: L \to gl(V)$ be a finite dim rep of a s.s. Lie alg L, then V is completely reducible.

Pf: $1 (\Leftrightarrow)$ $\forall$ submodule $W < V$, $\exists$

submodule W' s.t. $V = W \oplus W'$)

Pf: Induction on $\dim V$.

Case I. $\exists W < V$, $\dim V/W = 1$

(1). W is reducible

$\Rightarrow$ ① $\exists\, 0 \neq W' \subsetneq W \subsetneq V$

$\Rightarrow W/W' < V/W'$

$\dim {}^{V/W'}\!/_{W/W'} = 1$

By induction, $\exists\, W' \subseteq \hat{W} \subseteq V$ s.t.

$V/W' = W/W' \oplus \tilde{W}/W'$

② $\dim \tilde{W}/W' = 1$

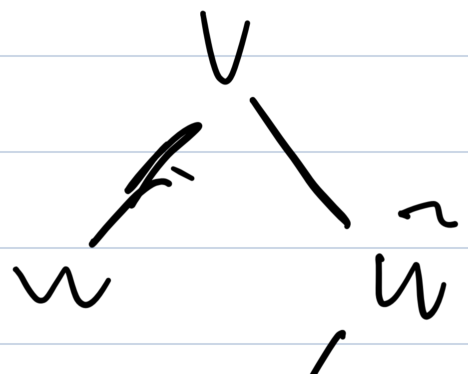
By induction.

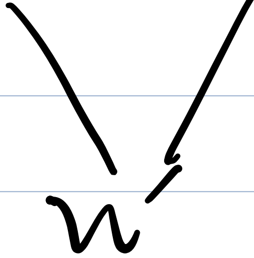
$\exists X < \tilde{W}$, s.t. $\tilde{W} = W' \oplus X$

$\dim X = 1$

$\dim X = \dim \tilde{W} - \dim W'$

③ $\dim X + \dim W = \dim V$.





$$\underline{X \cap W \subseteq W'}$$

$$W/W' \cap \tilde{W}/W' = 0$$

$$\Rightarrow X \cap W = 0$$

$$\Rightarrow V = W \oplus X$$

(2) W is irre.

Assume $\varphi : \underset{\substack{\shortparallel \\ \ker\varphi \oplus L'}}{L} \to gl(V)$

Let $C = C_\varphi(\beta)$ the Casimir element.

(*) $[C, \varphi(\beta)] = 0$, $\forall x \in L$

$\Rightarrow \forall v \in V$

$$C\varphi(x)(v) = \varphi(x)\, C(v)$$

$\Rightarrow C \in \mathrm{Hom}_L(V, V)$

$\Rightarrow \ker C < V$ is a submodule.

(*) $\dim V/W = 1$

Lemma 6.14. $\Rightarrow \mathcal{L}.V/W = 0$

$\Rightarrow \mathcal{L}.V \subseteq W$

$\forall x \in \mathcal{L}$, $\varphi(x)V \subseteq W$

$\Rightarrow C(V) \subseteq W$ $(C = \sum \varphi(x_i)\varphi(y_i))$.

$$C\,(\overset{n}{\overleftrightarrow{* \cdots}} \mid *) = (\overset{n}{\overleftrightarrow{* \cdots}} \mid *)\begin{pmatrix} * & * \\ \sim & \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \operatorname{tr} C = \operatorname{tr} C|_W = \dim \mathcal{L}' \neq 0.$

Since W is irre.

$$[c|_W, \varphi(x)|_W] = 0, \quad \forall x \in L$$

By schur's Lemma.

$$c|_W = \dim L / \underline{\dim W \cdot \mathrm{Id}_W}$$

$$C(V) \subseteq W$$

$\Rightarrow \mathrm{Im}\, C = W$

$\Rightarrow \dim \ker C = 1$ and

$\ker C \cap W = 0.$

$\Rightarrow V = \ker C \oplus W$

Case II, In general.

$0 \neq W < V$

$\Rightarrow$ $\mathrm{Hom}_{\mathbb{F}}(V, W)$ is $\mathcal{L}$-module

Define $V_0 = \{ f \in \mathrm{Hom}_{\mathbb{F}}(V, W) \mid f|_W \in \mathbb{F} \cdot \mathrm{Id}_W \}$

$\subseteq \mathrm{Hom}_{\mathbb{F}}(V, W)$

$\forall x \in \mathcal{L}$, $f \in \mathrm{Hom}_{\mathbb{F}}(V, W)$.

$(x.f)v = x.f(v) - f(x.v)$

If $f \in V_0$, $\forall w \in W$

$\Rightarrow (x.f)(w) = x.f(w) - f(x.w)$

$= \lambda x.w - \lambda x.w = 0.$

$\Rightarrow x.f \in V_0$

Hence $V_0 < \mathrm{Hom}_{\mathbb{F}}(V, W)$

$$V_1 \triangleq \{ f \in V_0 \mid f(W) = 0 \} < V_0$$

V_1 is an L-submodule.

Claim.

$$\dim V_0 / V_1 = 1$$

(By linear algebra).

By Case I. $\exists\, V_2 < V_0$ s.t.

$V_2 \oplus V_1 = V_0$

$\dim V_2 = 1 \Rightarrow \exists f \in V_0$

$V_2 = \mathbb{F}\underset{\sim}{f}$

$f\big|_W = \lambda \, Id_W \neq 0$

By Lemma 6.14.

$L . V_2 = 0 . \Rightarrow x . f(v) = f(x . v)$

$\Rightarrow$ $\ker f < V$ is a submodule

$\dim \ker f = \dim V - \dim W$

$(f|_W = \lambda Id_W \neq 0 , \ Im f \subseteq W).$

$\ker f \cap W = 0$

$\Rightarrow \quad V = \ker f \oplus W.$

§6.4. Presentation of.

Jordan decomposition.

$$\begin{tikzcd}[row sep=small, column sep=tiny] \mathcal{L} \text{ semisimple} \Rightarrow \mathcal{L} = \mathrm{ad}\,\mathcal{L} = & \mathrm{Der}(\mathcal{L}) \arrow[d, phantom, "\cap|"] \\ & \mathfrak{gl}(\mathcal{L}) \end{tikzcd}$$

$\forall\, x \in \mathcal{L}, \ \mathrm{ad}_x = \delta + \sigma$

$\delta . \sigma \in \mathrm{Der}(\mathcal{L}) \cong \mathcal{L}$

x_s ad-s.s. x_n-ad-nil.

$\Rightarrow \exists\, x_s, x_n \in L$, s.t.

$\underline{ad_x = ad\, x_s + ad\, x_n \quad [x_s, x_n] = 0}$.

$\underline{x = x_s + x_n}$ abstract Jordan decom.

Theorem. Let $L \subseteq gl(V)$ be s.s.

Then $\forall x \in L$, $\underline{x = x_s + x_n}$ is the usual Jordan decom of x in $gl(V)$

Pf: It is enough to show

$x = s + n$ the usual Jordan decom.,

we have $s, n \in L$

$\mathrm{ad}_x \in gl(L)$

Since $\mathrm{ad}_x(L) \subseteq L$ $\quad \underline{\mathrm{ad}_s = p(\mathrm{ad}_x)}$

$\underline{\mathrm{ad}_n = q(\mathrm{ad}_s)}$

(Prop 4.7).

$\Rightarrow \mathrm{ad}\, s(L) \subseteq L$

$\mathrm{ad}\, n(L) \subseteq L$

$\Rightarrow s, n \in N_{gl(V)}(L) \overset{\Delta}{=} N \quad ([L, N] \subseteq L)$

$L \triangleleft N$

Since L is s.s.

$\Rightarrow$ $\mathcal{L} \subseteq sl(V)$ (Lemma 6.14).

If W is an $\mathcal{L}$-submodule of V.

Define $\mathcal{L}_W = \{y \in gl(V) \mid y.W \subseteq W,$.
$\operatorname{tr} y|_W = 0\}$

$\mathcal{L} \subseteq \mathcal{L}_W$.

Claim: $\mathcal{L}_W$ is a Lie subalg

of $gl(V)$. ✓.

$\mathcal{L}_V = sl(V)$

Set $\mathcal{L}' = (\cap \mathcal{L}_W) \cap N$

is $\mathcal{L}$-submodule

In fact,

(1) $\mathcal{L}'$ is a subalg of $gl(V)$.

(2) $S, n \in \mathcal{L}'$

Claim: $\mathcal{L}' = \mathcal{L}$ (If this claim holds we are done).

(1) $\mathcal{L}'$ is an $\mathcal{L}$-module via

$$\mathrm{ad}_{gl(V)} \mathcal{L} : \mathcal{L}' \to \mathcal{L}'$$

(because $\mathcal{L} < \mathcal{L}'$ subalg).

By Weyl theorem.

$\mathcal{L}' = \mathcal{L} \oplus M$, M is a submodule of L'

$[L, L'] \subseteq L$

$\Rightarrow [L, M] = 0$

For $y \in M$ $[L, y] = 0$

For any irre. L-module $W < V$,

by Schur Lemma.

$y|_W = \lambda \, Id_W$, $\lambda \in F$

$y \in M \subseteq L' \subseteq L_W$

$\Rightarrow$ $tr \, y|_W = 0 \Rightarrow \lambda = 0$

$V = V_1 \oplus \cdots \oplus V_k$ V_i irre.

$\Rightarrow$ $y|_n = 0$

Cor. L is s.s. Lie alg.

$\varphi : L \to gl(V)$ rep.

$x = x_n + x_s$ abstract Jordan decomp.

of x

$\Rightarrow$ $\varphi(x) = \varphi(x_s) + \varphi(x_n)$ is the

usual decomp.

$sl_2(\mathbb{F})$.

Example. $V = \mathbb{F}[X, Y]$

$$V = \sum_{m=0}^{\infty} \left(\bigoplus_{i=0}^{m} \mathbb{F} X^{i} Y^{m-i} \right)$$

$$= \sum_{m=0}^{\infty} V(m)$$

$$\varphi : sl_2(\mathbb{F}) \to gl(V).$$

$$X \to X \frac{\partial}{\partial Y}$$

$$Y \to Y \frac{\partial}{\partial Y}$$

$$h \to [X\frac{\partial}{\partial Y}, Y\frac{\partial}{\partial Y}]$$

$$\shortparallel$$

$$X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$$

$\dim W(m) = m+1$

$h(X^i Y^j) = (i-j) X^i Y^j$

$\Rightarrow$ $W(m)$ is irr. !

It has weights

$m, m-2, \dots, -m$

$\not\Leftarrow$ it is irre.

(Because we can analyze

its component).

Example.

$\forall m \geq n \in \mathbb{Z}_{\geq 0}$

$W = V(m) \otimes V(n)$ is an L-module

$V(m) = \mathrm{Span}\{v_0, \cdots, v_m\}$

$V(n) = \mathrm{Span}\{w_0, \cdots, w_n\}$

$h v_i = (m-2i) v_i$

$h w_i = (n-2i) w_i$

$\{v_i \otimes w_j\}$ forms a basis.

$$h(v_i \otimes w_j) = h.v_i \otimes w_j + v_i \otimes h.w_j$$

$$= (m+n-2i-2j) v_i \otimes w_j$$

$W_{m+n-2i-2j}$.

The weights of W is

$\{m+n, \dots, -m-n\}$.

$$W \xrightarrow{\sim} V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(|m-n|)$$

If V is an irre. L-module

$\dim V < +\infty$

$\varphi : L \to gl(V)$ $\quad x, y, h$

$\Rightarrow$ $\varphi(x), \varphi(y) \in gl(V)$ are nilp.

$\Rightarrow$ $e^{\varphi(x)}, e^{\varphi(y)} \in G(V)$

$\tau \overset{\Delta}{=} e^{\varphi(x)} e^{\varphi(-y)} e^{\varphi(x)} \in GL(V)$

$\Rightarrow \tau \varphi(h) \tau^{-1} = -\varphi(h)$

$\forall v \in V_\lambda$ i.e. $hv = \lambda v$

$\varphi(h)(v) = \lambda v$

$\varphi(h)(\tau(v)) = -\tau\varphi(h)v = -\lambda\tau v$

$\Rightarrow \tau v \in V_{-\lambda}$.

$V_\lambda \neq 0 \Leftrightarrow V_{-\lambda} \neq 0$.

$\dim V_\lambda = \dim V_{-\lambda}$.

$$W = F[X^{\pm 1}, Y^{\pm 1}]$$

$$= \bigoplus_{m=-\infty}^{+\infty} W(m)$$

$$W(m) = \bigoplus_{i=-\infty}^{+\infty} F X^i Y^{m-i}$$

$$X \to X \frac{\partial}{\partial Y}$$

$$Y \to Y \frac{\partial}{\partial X}$$

$h \to X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$

In $V(m)$

$$W_m = \bigoplus_{i=-\infty}^{m} \mathbb{F}X^i Y^{m-i} \subseteq V$$

submodule

$m<0$, W_m irre.

If $m \geq 0$

$$W_m' \overset{\delta}{=} \bigoplus_{i=0}^{m} \mathbb{F}X^i Y^{m-i} \text{ is irre.}$$

(**)

$$sl_2(\mathbb{F}) \hookrightarrow sl_4(\mathbb{F})$$

$$sl_2(\mathbb{F}) \hookrightarrow gl(sl_4(\mathbb{F}))$$

$$x \to ad_{sl_4}(x).$$

$$h(e_{ij}) = [h, e_{ij}]$$

$$= (\delta_{i1} - \delta_{j1} - \delta_{i2} + \delta_{j2}) e_{ij}$$

$$= \lambda_{ij} e_{ij}$$

$$\lambda_{ij} = 2 \iff e_{ij} = e_{12}$$

$$\lambda_{ij} = 2^{-1} \iff e_{ij} = e_{21}$$

$$sl_4(\mathbb{F}) \xrightarrow{\sim} V(2) \oplus \,.$$

§8. Root space Decom.

(*) $L \neq 0$ s.s.

$\Rightarrow L$ is not nilp.

By Engel's thm.

$\exists\ x = x_s + x_n$. abstract Jordan.

$x_s \neq 0$.

Define 8.1.

A subalg of L is called toral if $\forall x \in T$, $\mathrm{ad}_L x$ is s.s.

If $L \neq 0$ s.s. $\Rightarrow \exists T \neq 0$ toral.

x s.s. $\Rightarrow \mathbb{F}x$ ✓.

§8.1. maximal toral and roots.

Lemma 8.1.

If $T \subseteq L$ is a toral

$\Rightarrow [T,T] = 0$

pf: T is toral. It's enough

to show $\mathrm{ad}_T x = 0$

$\mathrm{ad}_L x$ s.s. $\Rightarrow$ $\mathrm{ad}_T x$ s.s.

$\exists\, y_1 \sim y_k \in T$ basis s.t.

$\mathrm{ad}_T x$ diag.

If $\mathrm{ad}_T x \neq 0 \Rightarrow \exists\, 0 \neq y \in T$,

$$\left.\begin{array}{l} a \neq 0 \text{ such that } \mathrm{ad}_x(y) = ay \\ \Leftrightarrow \mathrm{ad}_y(x) = -ay \end{array}\right\} \star .$$

$y \in T \Rightarrow \mathrm{ad}_T y$ s.s.

$\mathrm{ad}\,y(y)=0$ $\exists$ a basis of T.

$\{y, v_2, \cdots, v_k\}$ s.t.

$\mathrm{ad}_T\, y$ is diag.

$\mathrm{ad}_y(v_i) = a_i v_i$

$$x = \sum_{i=2}^{k} b_i v_i + b_1 y$$

$$0 \neq -ay = \mathrm{ad}\,y\left(\sum b_i v_i + b_1 y\right)$$

$$= \sum_{i=2}^{n-2} a_i b_i v_i$$

$\Rightarrow$ $a=0$, contradiction!

$\square$

* Fix a maximal toral subalg $H \Rightarrow [H,H]=0$

$\{ad_L \cdot h \mid h \in H\}$ is a family of linear transformation wich are diagonalizable

$$[ad_L h_1, ad_L h_2]=0.$$

$\Rightarrow \exists \{x_1, \cdots, x_n\}$ basis of L.

s.t. $\mathrm{ad}_L h$ is diagonal, $\forall h \in H$.

$H = \mathrm{Span}\{h_1, \dots, h_k\}$ basis

$$h_i x_j = \lambda_{ij} x_j$$

Define $f_j \in H^*$, $f_j(h_i) = \lambda_{ij}$

$\Rightarrow \forall h \in H$, $h x_j = f_j(h) x_j \quad \forall j$.

$$h_i x_j = f_j(h_i) x_j, \ \forall i, j$$

$$\Rightarrow \quad x_i \in L_{f_j} = \{x \in L \mid [h,x] = f_j(h)x, \ \forall h \in H\}$$

$\forall \alpha \in H^* = \mathrm{Hom}(H, \mathbb{F})$, Define

$$L_\alpha \overset{\Delta}{=} \{x \in L \mid [h,x] = \alpha(h)x, \forall h \in H\}$$

Then $L = \bigoplus_{\alpha \in H^*} L_\alpha$

$$\Phi = \{\alpha \in H^* \setminus \{0\} \mid L_\alpha \neq 0\}.$$

$$\Rightarrow L = L_0 \oplus \left(\bigoplus_{\alpha \in \Phi} L_\alpha\right)$$

Φ is called root system of L.

$\alpha \in \Phi$ is called a root.

$L = L_0 \oplus \left(\bigoplus_{\alpha \in \Phi} L_\alpha\right)$ is called

the Cartan decomposition.

Claim:

- $\mathcal{L}_0 = H$

- $\forall \alpha \in \Phi$, $\dim \mathcal{L}_\alpha = 1$.

- $\mathrm{Span}\, \Phi = H^*$.

- $K(\mathcal{L}_\alpha, \mathcal{L}_\beta) = 0$, $\alpha + \beta \neq 0$

  $\Rightarrow \alpha \in \Phi \Leftrightarrow -\alpha \in \Phi$.

Prop. 8.5.

1) $\forall \alpha, \beta \in H^*$

$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

(2) If $x \in L_\alpha$, $\alpha \neq 0$, then

$\mathrm{ad}_L x$ is nilp.

(3) $\forall \alpha, \beta \in H^*$, If $\alpha+\beta \neq 0$

$\Rightarrow$ $K(L_\alpha, L_\beta) = 0$

pf: $\forall x \in L_\alpha$, $y \in L_\beta$

$\forall h \in H$, then

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]]$$

$$= \alpha(h)[x,y] + \beta(h)[x,y]$$

$$= (\alpha(h)+\beta(h))[x,y].$$

(2) $L = L_0 \oplus \sum_{\alpha\in\Phi} L_\alpha$, $|\Phi| < +\infty$

$\forall\, 0 \neq \alpha \in H^*$

$\Rightarrow \exists N \in \mathbb{Z}_{\geq 0}$, s.t. $\forall \beta \in \Phi \cup \{0\}$

$n\alpha + \beta \notin \Phi \cup \{0\}$, $\forall n \geq N$

$$\Rightarrow \forall y \in L,\ (\operatorname{ad} x)^n \underset{\overset{\uparrow}{L_{n\alpha+\beta}}}{y} = 0.$$

$\Rightarrow \quad (\mathrm{ad}\,x)^n = 0$

(3) $\quad \alpha+\beta \neq 0$

$\Rightarrow \exists h \in H, \quad (\alpha+\beta)(h) \neq 0.$

$\forall x \in L_\alpha, \ y \in L_\beta$

$K(h, [x,y]) = K([h,x], y)$

$= \alpha(h) K(x,y)$

LHS $= -K(h, [y,x])$

$= - \beta(h) K(x,y)$

$\Rightarrow K(x,y) = 0.$

$\square$

Cor. 8.6.

$L_0 = C_L(H)$ is a subalg of L

Then $K|_{L_0}$ is non-deg.

Pf: If $z \in L_0$, $K(z, L_0) = 0$

$\forall x, y \in L_0$

$K|_{L_0}(x, y) = \mathrm{tr}\, \mathrm{ad}_L x \, \mathrm{ad}_L y$

By Prop 8.5 (3)

$\Rightarrow \forall \alpha \in \Phi, z \in L_0$

$K(z, L_\alpha) = 0.$

$\Rightarrow K(z, L) = 0$

By K is non deg

$\Rightarrow z = 0$

Lemma 8.7.

If $f, g \in gl(V)$, $\dim V < +\infty$

$f^n = 0$, $fg = gf$

$\Rightarrow fg$ is nilp.

Prop 8.8.

Let $H \le L$ be a maximal toral subalg. Denote $C = \underset{L_0.}{\underset{\shortparallel}{C_L(H)}}$, then $C = H$.

Pf: (1) $\forall x \in C$.

$x = x_s + x_n$

(1) Claim: $x_s, x_n \in C$

Pf: $\operatorname{ad} x\,(H) = 0$

Prop 4.7 (c) $\Rightarrow$ $\operatorname{ad} x_s\,(H) = 0$

$\operatorname{ad} x_n\,(H) = 0$

$\Rightarrow$ $x_s, x_n \in C_L(H)$.

(2) Claim: $\forall$ $x \in C$.

If x is s.s.

$\Rightarrow$ $x \in H$.

Pf: $[x, H] = 0$.

$\Rightarrow$ $H + Fx$ is a subalg.

$\forall$ $x, y \in H$ $[x, y] = 0$

$x + y$ is s.s.

$\Rightarrow$ $H + Fx$ is a toral

$\Rightarrow \quad H + Fx = H.$

(3) Claim: $K|_H$ is non deg.

$K(h, H) = 0$

$\forall \alpha \in \Phi. \quad h \in H \subseteq C_L(H) = L_0$

$\Rightarrow K(h, L_\alpha) = 0.$

$\forall x \in C \qquad x = x_s + x_n$

$x_s \in H \quad x_n \in C$

$[x_n, h] = 0$

$K(h, x_n) = tr(ad_L h \, ad_L x_n) = 0.$

(Lemma 8.7).

$\Rightarrow K(h, L) = 0 \Rightarrow h = 0$

(4)

Claim: C is nilp. $\forall x \in C$

$(x = x_s + x_n, \; x_s \in H)$

$ad_C x_s(y) = [x_s, y] = 0 \quad \forall y \in C$

$\Rightarrow ad_C x$ nilp.

Engel ✓.

(5) Claim:

$H \cap [C, C] = 0$

Pf: $C = C_{L(H)}$

$\Rightarrow [H, C] = 0$

$K(H, [C, C]) = 0.$

$K|_H$ non deg $\Rightarrow [C, C] \cap H = 0.$

(b) Claim: $[C, C] = 0$

(c) Claim: $C = H$

Corollary 8.9.

$K|_H$ is non deg

Remark 8.10.

$\{ K(h, \cdot) \mid h \in H \} = H^*$

$\forall \alpha \in H^*$, $\exists! t_\alpha$, $K(t_\alpha, \cdot) = \alpha$

§8.3. Orthogonality properties.

Prop 8.11.

(a) $\operatorname{Span} \Phi = H^*$.

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$

$\Rightarrow \Phi = -\Phi$

(c). $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$

$\Rightarrow [x, y] = K(x, y) t_\alpha$

(d) $\dim [L_\alpha, L_{-\alpha}] = 1$

(e) $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0.$

(f). If $\alpha \in \Phi$. $0 \neq x_\alpha \in L_\alpha$

$\exists\, y_\alpha \in L_{-\alpha}$ s.t.

$$\operatorname{Span}_{\mathbb{F}} \{ x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha] \}$$

$$\downarrow \wr$$

$$sl_2(\mathbb{F}).$$

(g) $h_\alpha = \dfrac{2 t_\alpha}{K(t_\alpha, t_\alpha)}$, $h_\alpha = -h_{-\alpha}$

Pf: (a)

If $\operatorname{Span} \Phi \subsetneq H^*$, $\Rightarrow \exists\, h \in H$, s.t.

$\forall \alpha \in \Phi, \ \alpha(h) = 0.$

$\forall x \in L_\alpha$

$[h, x] = \alpha(h) x = 0$

$\Rightarrow [h, L] = 0, \quad \times.$

(b) $K(L_\alpha, L_\beta) = 0 \quad \alpha + \beta \neq 0.$

If $-\alpha \notin \Phi.$ $K(L_\alpha, L) = 0, \quad \times.$

(c) $K(h, [x, y]) = \alpha(h) K(x, y)$

$= K(t_\alpha, h) K(x, y)$

$$= K(h, K(x,y)t_\alpha)$$

$$\Rightarrow [x,y] = K(x,y)\, t_\alpha$$

(By the non-degenerated property).

(b) $\forall\ 0 \neq x \in \mathcal{L}_\alpha$

if $K(x, \mathcal{L}_{-\alpha}) = 0$

$\Rightarrow K(x, \mathcal{L}) = 0$, ✗

$\Rightarrow K(\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}) \geqslant 1$

(c) $\Rightarrow K(\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}) = 1$.

(e). $\alpha(t_\alpha) = K(t_\alpha, t_\alpha)$: ✓.

If $K(t_\alpha, t_\alpha) = 0$

$$\Rightarrow \underset{\alpha(L_\alpha)\, t_\alpha}{\underset{\shortparallel}{[t_\alpha, L_\alpha]}} = 0$$

$$\Rightarrow [t_\alpha, L_{-\alpha}] = 0.$$

By (d), $\exists\ x \in L_\alpha,\ y \in L_{-\alpha}$

$K(x, y) \neq 0$

$[x, y] = K(x, y)\, t_\alpha$

$$\Leftrightarrow \left[x, \boxed{\frac{y}{K(x,y)}}\right] = t_\alpha$$

x substitute y.

$S = \mathrm{Span}\{x, y, t_\alpha\}$

$[x, y] = t_\alpha$

$[t_\alpha, x] = 0$

$[t_\alpha, y] = 0$

$\Rightarrow S$ is solvable.

$\mathrm{ad}_L : L \to gl(L)$

$\mathrm{ad}_L(S) \subseteq gl(L)$ solvable

$\Rightarrow \forall s \in [S, S] = \mathbb{F} t_\alpha$

(Cor 4.2) $\Rightarrow$ $\mathrm{ad}_L S$ is nilp.

$\Rightarrow$ $\mathrm{ad}_L t_\alpha$ nilp, s.s.

$\Rightarrow$ $t_\alpha = 0$, X.

(f). $\exists$ $x_\alpha \in L_\alpha$

$y_\alpha \in L_{-\alpha}$

$$K(x_\alpha, y_\alpha) = \frac{2}{K(t_\alpha, t_\alpha)}$$

$$h_\alpha = [x_\alpha, y_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)} t_\alpha$$

$\Rightarrow$ $[h_\alpha, x_\alpha] = 2x_\alpha$

$[h_\alpha, y_\alpha] = -2 y_\alpha$

$x_\alpha \to x \quad y_\alpha \to y \quad h_\alpha \to h$

$\Rightarrow S_\alpha = \mathrm{Span}_{\mathbb{F}}\{x_\alpha, y_\alpha, h_\alpha\} \xrightarrow{\sim} sl_2(\mathbb{F})$

(g) ✓.

$S_\alpha \leq L$ subalg.

$\Rightarrow$ L is an $sl_2(\mathbb{F})$-module.

Prop 8.12.

(a) $\forall \alpha \in \Phi$ $\dim L_\alpha = 1$

In particular, $S_\alpha = L_\alpha + L_{-\alpha} + \mathbb{F} t_\alpha$

$\mathbb{F} t_\alpha = H_\alpha = [L_\alpha, L_{-\alpha}]$

(b) If $\alpha \in \Phi$, $c\alpha \in \Phi$

$\Leftrightarrow c = \pm 1$.

$$V = \underset{\overset{\shortparallel}{V_0.}}{H} + \bigoplus_{c \in \mathbb{F}^*} L_{c\alpha} = \bigoplus_{c \in \mathbb{F}} L_{c\alpha}$$

V is a $sl_2(\mathbb{F})$-module

$\ker\alpha$, $\mathbb{F} h_\alpha$

$\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$

$\Rightarrow \frac{1}{2}\alpha \notin \Phi$.

$h_\alpha L_{c\alpha} = 2c L_{c\alpha}$

(c) If $\alpha, \beta \in \Phi \Rightarrow \beta(h_\alpha) \in \mathbb{Z}$, and $\beta - \beta(h_\alpha)\alpha \in \Phi$

"$\beta(h_\alpha)$ is Cartan integer"

(d) $[L_\alpha, L_\beta] = L_{\alpha+\beta}$

(e) If $\alpha, \beta \in \Phi$

$\beta \neq \pm\alpha$, Let r, q be the

Cartan integers,

$\beta - r\alpha, \beta + q\alpha \in \Phi$

$\Rightarrow \beta + i\alpha \in \Phi, \quad -r \leq i \leq q$

$\beta(h_\alpha) = r - q$.

(f)

$L = \langle L_\alpha, L_{-\alpha} \mid \alpha \in \Phi \rangle$

Pf: (c)

for $\beta = \pm\alpha$, ✓

If $\beta \neq \pm\alpha$

$$M \triangleq \bigoplus_{i \in \mathbb{Z}} L_{\beta + i\alpha}$$

$\beta \pm i\alpha \neq 0$, $\forall i \in \mathbb{Z}$

$S_\alpha = L_\alpha + L_{-\alpha} + \mathbb{F} t_\alpha \subseteq L_\alpha + L_{-\alpha} + L_0$

M is a S_α-module.

$\forall x \in L_{\beta+i\alpha}$

$$[h_\alpha, x] = (\beta + i\alpha)(h_\alpha)\, x$$

$$= (\beta(h_\alpha) + 2i)\, x$$

If $\beta + 2i\alpha \in \Phi$ (take $i=0$).

$\dim M < +\infty \Rightarrow \beta(h_\alpha) + 2i \in \mathbb{Z}$

② $i \neq j$ $\underbrace{\dim L_{\beta+i\alpha} = 1}_{\beta+2i \not\equiv \beta+2j \bmod 2}$, if $\beta + i\alpha \in \Phi$

$\neq$ The weights of M are all

even or all odd

$\Rightarrow$ M is irre. sl_2 module

By the definiton of r and q

we know the highest is

$\beta(h_\alpha)+2q$

lowest $\beta(h_\alpha)-2r$

$$\Rightarrow \quad \beta(h_\alpha)-2r=-(\beta(h_\alpha)+2q)$$

$$\Rightarrow \quad \beta(h_\alpha)=r-q$$

§8.4. Rationality property

L s.s. Lie alg $/ \mathbb{C}$

Take a maximal toral H.

Cartan decom.

$$L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$$

$$H_1 \quad L = H_1 \oplus \sum_{\beta \in \Phi_1} L_\beta$$

$\forall \alpha \in \Phi$, $\exists !$ $t_\alpha \in H$

$\alpha(h) = K(t_\alpha, h)$

$K\big|_H$ non-deg

Define $(\gamma, \delta) \overset{\Delta}{=} K(t_\gamma, t_\beta)$

$\gamma, \delta \in H^*$

$t_{\lambda\gamma + \mu\delta} = \lambda t_\gamma + \mu t_\beta$

$$\beta(h_\alpha) = \beta\left(\frac{2t_\alpha}{K(t_\alpha, t_\alpha)}\right)$$

$$= \frac{2\beta(t_\alpha)}{K(t_\alpha, t_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

Let $\alpha_1 \sim \alpha_l \in \Phi$

is a basis of H^*

If $\beta \in \Phi \Rightarrow \beta = \sum_{i=1}^{l} c_i \alpha_i$

$$\Rightarrow \frac{2(\beta,\alpha_j)}{(\alpha_j,\alpha_j)} = \sum_i C_i \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$$

K is nondeg $\Rightarrow$ ( ) is non deg

$$\frac{2(\beta,\alpha_j)}{(\alpha_j,\alpha_j)} \in \mathbb{Z}$$

$$\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} \in \mathbb{Z} \Rightarrow \underbrace{C_i \in \mathbb{Q}.}$$

$\Phi \subseteq \underline{\mathrm{Span}_{\varphi}\{\alpha_1, \cdots, \alpha_l\}}$

Let $E_\varphi = \mathrm{Span}_\varphi\{\alpha \mid \alpha \in \Phi\}$

is a vector space $/\ \varphi$

$\Rightarrow \dim E_\varphi = \dim H = l$

$\forall \lambda, \mu \in H^*$

$(\lambda, \mu) = \mathrm{Tr}\ \mathrm{ad}_{t_\lambda} \mathrm{ad}_{t_\mu}$

$[t_\lambda . H] = 0$

$[t_\lambda, L_\alpha] = \alpha(t_\lambda) L_\alpha$

$$= \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu)$$

$\Rightarrow$ $\beta\in\Phi$. $\kappa(t_\alpha, t_\beta) = (\alpha,\beta)$

$\|$

$$(\beta,\beta) = \sum_{\alpha\in\Phi} \alpha(t_\beta)\alpha(t_\beta)$$

$$= \sum_{\alpha\in\Phi} (\alpha,\beta)^2$$

$$\Leftrightarrow \frac{4}{(\beta,\beta)} = \sum_{\alpha\in\Phi}\left(\frac{2(\alpha,\beta)^2}{(\beta,\beta)}\right)^2$$

$\Downarrow$

丞 我嘞个强死 哼.

$$\overset{\downarrow .}{\Rightarrow \ (\alpha,\beta)\in\mathbb{Q}}\ ,\ \forall\alpha,\beta\in\Phi$$

$$(\beta,\beta)=\sum_{\alpha\in\Phi}(\alpha,\beta)^2$$

$$\geq 2(\beta,\beta)^2$$

$\forall\lambda\in E_{\mathbb{Q}}$

$$(\lambda,\lambda)=\sum_{\alpha\in\Phi}\underset{\overset{\uparrow}{\mathbb{Q}}}{(\alpha,\lambda)}^2\geq 0$$

$\Rightarrow$ $(\ ,\ )$ is a positive definite sym. bilinear form on $E_{\mathbb{Q}}$

$E \triangleq E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ Euclidean space

$(\, , \,) : E \times E \to \mathbb{R}$

$\Phi \subseteq E$ $\dim E = \ell = \dim H.$

Theorem 8.13

L s.s. Lie alg. over $\mathbb{F}$.

$\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} = 0$.

H is a maximal toral subalg.

$\Phi \subseteq H^*$ is the set of roots

$$L = C_L(H) \oplus \sum_{\alpha \in H} L_\alpha$$

$$E = \mathbb{R} \otimes_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} \Phi$$

then

(a) Φ spans E, $0 \notin \Phi$

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and

$c\alpha \in \Phi \Leftrightarrow c = \pm 1$

(c) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$

(d) $\alpha, \beta \in \Phi$ $\quad \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

(Root system).

$(L, H) \to (E, \Phi)$

Use Root System to

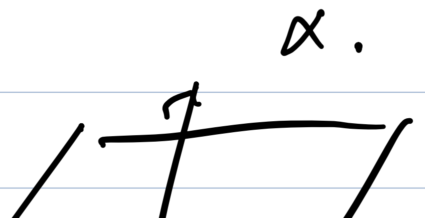
Classify s.s. alg.

Chapter III. Root systems.

§9. Axiomatics.

§9.1. Reflection.

E a fixed Euclidean space

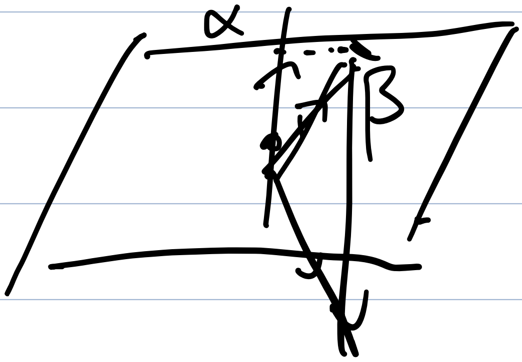


$$(\ , \ ) : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$$

inner product.

(1). $P_\alpha = \alpha^\perp = \left\{ x \in E \,\middle|\, (x,\alpha) = 0 \right\}$

(2) $\sigma_\alpha(\beta) = \beta - 2 \dfrac{(\beta,\alpha)}{(\alpha,\alpha)} \alpha$



(3) $(\sigma_\alpha(\beta), \sigma_\alpha(\gamma)) = (\beta, \gamma)$

(4) $\langle \beta , \alpha \rangle = \dfrac{2(\beta,\alpha)}{}$

(α,α)

(5). $\sigma_\alpha^2 = \mathrm{Id}$ $\quad \sigma_\alpha = \sigma_{\lambda\alpha}$

Lemma. 9.1.

Let $\Phi \subseteq E$, $|\Phi| < +\infty$

$\mathrm{Span}_{\mathbb{R}} \Phi = E$

$\forall \alpha \in \Phi \quad \sigma_\alpha(\Phi) = \Phi \; [\Phi_\alpha \in S_\alpha)$

If $\sigma \in GL(E)$.

$\sigma(\Phi) = \Phi$, $\exists$ hyperplane P

s.t. $\sigma|_P = \mathrm{Id}_P$

$\exists\, 0 \neq \alpha \in \Phi$, $\sigma(\alpha) = -\alpha$

$\Rightarrow \quad \sigma = \sigma_\alpha$

pf: Set $\tau = \sigma\sigma_\alpha \in GL(E)$

$\sigma\sigma_\alpha \in S_\Phi$.

$\Rightarrow \quad E = \mathbb{R}\alpha \oplus P_\alpha$

$= \mathbb{R}\alpha \oplus P$

Φ Spans E, take $\{\alpha, \alpha_2, \cdots, \alpha_n\}$

be a basis of E.

$\alpha_i = \beta_i + b_i\alpha = \nu_i + a_i\alpha$

$\beta_i \in P_\alpha$, $\nu_i \in P$.

$$\tau(\alpha_i) = \sigma\sigma_\alpha(\beta_i + b_i\alpha)$$

$$= \sigma(\beta_i - b_i\alpha)$$

$$= \sigma(\gamma_i + a_i\alpha - 2b_i\alpha)$$

$$= \gamma_i + (2b_i - a_i)\alpha$$

$$= \alpha_i + 2(b_i - a_i)\alpha$$

$$\Rightarrow \tau(\alpha, \dots, \alpha_l) = (\alpha, \dots, \alpha_l)\begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & 0 & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

$$\tau \in S(\Phi)$$

$$\Rightarrow \exists k,\ \tau^k = Id_E.$$

$$\begin{array}{ccc} & & v \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}^{k} = \begin{array}{ccc} 1 & & kv \\ \hline & & 0 \\ & \ddots & \\ & & 1 \end{array}$$

$\Rightarrow v = 0$

$\Rightarrow \tau = \mathrm{Id}.$

§9.2. Root System.

Definition. 9.2

$\Phi \subseteq E$ is called a root system in E, if

(R1) $|\Phi| < +\infty$, Φ spans E, $0 \notin \Phi$.

(R2). $\alpha \in \Phi \Leftrightarrow -\alpha \in \Phi$

$c\alpha \in \Phi \Leftrightarrow c = \pm 1.$

(R3) $\forall \alpha \in \Phi,$

$\sigma_\alpha(\Phi) = \Phi.$

(R4) $\forall \alpha, \beta \in \Phi.$

$\langle \beta, \alpha \rangle \in \mathbb{Z}$

Remark. 9.3.

(1) $\Phi = -\Phi.$ $2 \mid |\Phi|.$

(2). (R1)(R3)(R4) $\Longrightarrow$ "Root System".

Exercise 9.

(3) $(E, (\ ,\ ))$ Euclidean.

$r>0$ $(E, r(\cdot,\cdot))$ <u>Euclidean.</u>

$$\sigma_\alpha(\beta) = \text{``}\sigma_\alpha(\beta)).\text{''}$$

Definition. (Weyl group)

$$\sigma_\alpha \in S_{\Phi}.$$

Φ root system.

Weyl group W of Φ

$$W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle \subseteq GL(E) \cap S(\Phi).$$

Lemma. 9.5.

If $\sigma \in GL(E)$.

$\Phi \subseteq E$ root system in E.

$\sigma(\Phi) = \Phi$, then

(1) $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)} \in W$.

(2) $\langle \sigma(\beta), \sigma(\alpha) \rangle = \langle \beta, \alpha \rangle$

pf. (1) $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\alpha)) = -\sigma(\alpha)$

$\forall x \in P_\alpha$

$\sigma \sigma_\alpha \sigma^{-1}(\sigma(x)) = \sigma(x)$

Lemma 9.1.

(2) $\forall \beta \in \Phi$.

$$\sigma_{\sigma(\alpha)}\sigma(\beta) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$$

$$\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta))$$

$$= \sigma(\beta - \langle \beta, \alpha \rangle \alpha)$$

$$= \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$$

$$\Rightarrow \langle \sigma(\beta), \sigma(\alpha) \rangle = \langle \beta, \alpha \rangle$$

Definition 9.6.

(Φ, E), (Φ', E') is called isomorphic, if $\exists \varphi : E \to E'$ iso. of vector spaces, s.t.

$\varphi(\Phi) = \Phi'$, and $\begin{array}{c} <\varphi(\beta), \varphi(\alpha)> \\ \parallel \\ <\beta, \alpha> \end{array}$,

$\forall \alpha, \beta \in \Phi$.

(*). If $\varphi : E \xrightarrow{\sim} E'$.

$\varphi(\Phi) = \Phi'$

$\tau \triangleq \varphi \sigma_\alpha \varphi^{-1} \in GL(E')$

$\tau(\Phi') = \Phi'$

$\tau = \sigma_{\varphi(\alpha)}$. $\tau(\varphi(\beta)) = \varphi(\beta) - <\varphi(\beta), \varphi(\alpha)> \varphi(\alpha)$

$\|$

$\varphi\sigma_\alpha(\beta) = \varphi(\beta) - \langle \beta, \alpha \rangle \varphi(\alpha).$

Apply to $\varphi(\beta)$

$\Downarrow$

$\Rightarrow \langle \varphi(\beta), \varphi(\alpha) \rangle = \langle \beta, \alpha \rangle$

$$\mathrm{Aut}(\Phi) = \{ \sigma \in GL(E) \mid \sigma(\Phi) = \Phi \}.$$

Definition. 9.7.

$\Phi \subseteq E.$

$\forall \alpha \in \Phi. \quad \alpha^\vee \triangleq \dfrac{2\alpha}{(\alpha,\alpha)} \in E.$

$$\Phi^\vee \triangleq \{ \alpha^\vee \mid \alpha \in \Phi \}$$

Dual of Φ.

Claim: $\Phi^\vee \subseteq E$ is a root system.

$$\langle \beta^\vee, \alpha^\vee \rangle = \langle \alpha, \beta \rangle \in \mathbb{Z}.$$

Remark. (1) $W_{\Phi^\vee} \cong W_\Phi$.

$$\sigma_{\alpha^\vee} = \sigma_\alpha \quad , \forall \alpha \in \Phi .$$

(2)

Lie algebra

$$\begin{tikzcd} (L, H) \arrow[r] & (E, \Phi) \\ t_\alpha & \alpha \arrow[l, mapsto] \\ h_\alpha & \alpha^\vee . \arrow[l, mapsto] \end{tikzcd}$$

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \longmapsto \frac{2t_\alpha}{(\alpha,\alpha)} = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)} = h_\alpha$$

§9.3. Pairs of Roots.

(R4). $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

$$(\beta,\beta) \qquad \mathbb{Z}.$$

$$\frac{4(\beta,\alpha)^2}{\|\alpha\|^2\|\beta\|^2} \leq 4. \qquad 0,1,2,3,4.$$

$$\text{“}=\text{”} \Leftrightarrow \beta = \pm\alpha.$$

$$(\beta,\alpha) = \|\beta\|\|\alpha\|\cos\theta.$$

$$\cos^2\theta = 1 \Leftrightarrow \beta = \pm\alpha$$

Or

$$0 \leq \langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = 4\cos^2\theta \leq 3.$$

Suppose $|\langle\alpha,\beta\rangle| \leq |\langle\beta,\alpha\rangle|$

$\Rightarrow \langle \alpha, \beta \rangle = 0, \pm 1.$

Set $\langle \beta, \alpha \rangle = k \quad |k| \leq 3.$

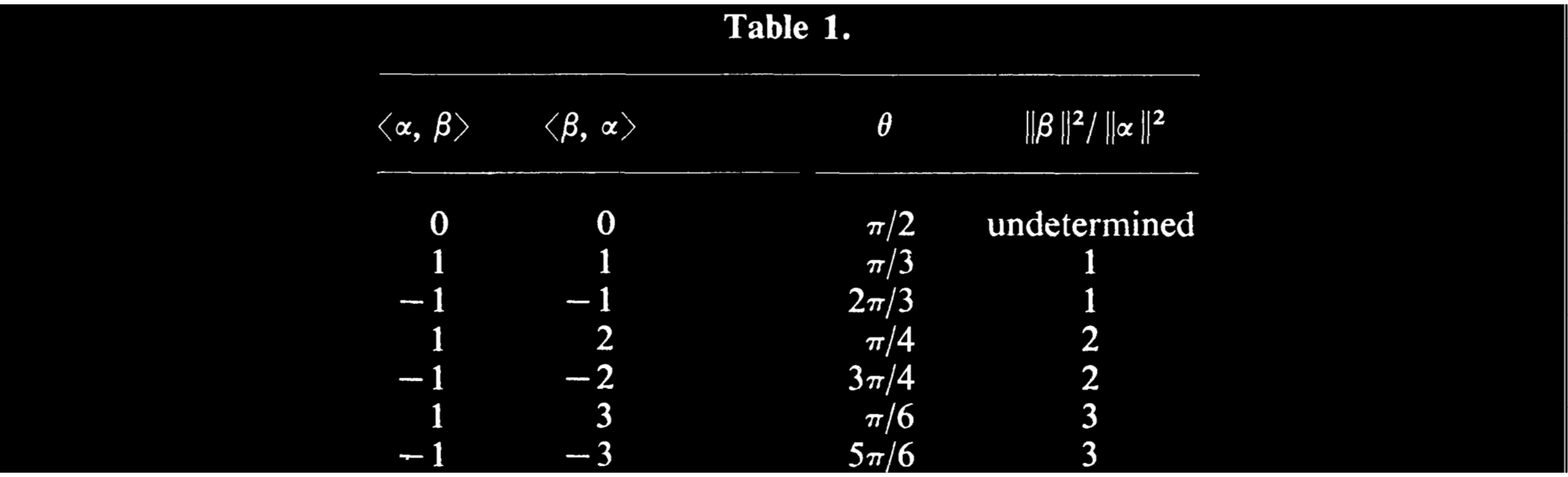


Table 1.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2 / \|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Lemma 9.8.

$\alpha, \beta \in \Phi$.

$\beta \neq \pm \alpha$.

(1) If $(\alpha, \beta) > 0 \Rightarrow \beta - \alpha \in \Phi$.

$\Updownarrow$.

(2) If $(\alpha, \beta) < 0 \Rightarrow \beta + \alpha \in \Phi$.

Pf: $(\alpha,\beta) > 0 \Rightarrow \langle\alpha,\beta\rangle, \langle\beta,\alpha\rangle > 0.$

$\Rightarrow \underbrace{\langle\alpha,\beta\rangle = 1}\checkmark.$ or $\langle\beta,\alpha\rangle = 1.$

$\Downarrow$

$$\sigma_{\beta}(\alpha) = \alpha - \beta \in \Phi.$$

$\square$

Application 9.9.

$\alpha, \beta \in \Phi.$ $\beta \neq \pm\alpha$

$M = \{\beta + i\alpha \mid i \in \mathbb{Z}\} \cap \Phi.$

$r, q \in \mathbb{Z}_{\geq 0}$ largest integers such that

$$\beta - r\alpha, \ \beta + q\alpha \in \Phi.$$

(1) Claim:

$$\forall -r \le i \le q$$

$$\beta + i\alpha \in \Phi.$$

pf: Otherwise, $\exists -r < i < q$

$$\beta + i\alpha \notin \Phi.$$

$\Rightarrow \exists -r \le p < s \le q$ s.t.

$$\underbrace{\beta + p\alpha \in \Phi, \quad \beta + (p+1)\alpha \notin \Phi}$$

$$\underline{\beta + (s-1)\alpha \notin \Phi \qquad \beta + s\alpha \in \Phi.}$$

$$(\beta + p\alpha, \alpha) \geq 0.$$

$$(\beta + s\alpha, \alpha) \leq 0.$$

$$\Rightarrow \quad (\beta, \alpha) \geq -p(\alpha, \alpha)$$

$$(\beta, \alpha) \leq -s(\alpha, \alpha).$$

Contradiction!

$$\Rightarrow \quad M = \{\beta + i\alpha \mid -r \leq i \leq q\}.$$

α - String through β.

(2) Claim:

$$\underset{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}{\underset{\parallel}{<\beta,\alpha>}} = r-q \quad (\beta(h_\alpha) = r-q).$$

$$\sigma_\alpha(\beta+q\alpha) = \beta - <\beta,\alpha>\alpha - q\alpha$$

$$\sigma_\alpha(\beta-r\alpha) = \beta - <\beta,\alpha>\alpha + r\alpha$$

$\Rightarrow$ σ_α reverse the string.

$\Rightarrow$ $\beta - <\beta,\alpha>\alpha - q\alpha = \beta - r\alpha$

$\Rightarrow \quad \langle \beta, \alpha \rangle = r - q.$

(3). Claim: $|M| \leq 4.$

$q + r = \langle \beta + q\alpha, \alpha \rangle \leq 3.$

$\Rightarrow \quad |M| = q + r + 1 \leq 4.$

§9.4. Examples.

$\dim E = 1$, $\Phi = \{\alpha, -\alpha\}$

Def. (Φ, E) $\dim E = \ell$,

We call the root system is

of rank ℓ

$sl_2(\mathbb{F}) = \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}h.$

$\Phi = \{\alpha, -\alpha\}$

$\alpha(h) = 2.$

---

(1) L s.s. H maximal toral

Φ root system. $\Phi = \{\alpha, -\alpha\}$

$\Rightarrow L \cong sl_2(\mathbb{F}).$ $W = S_2$

(2) rank=2.

Let $\theta = \mathrm{Max}\{\arccos \frac{(\alpha,\beta)}{\|\alpha\|\|\beta\|}\}.$

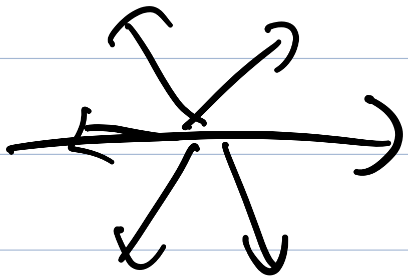
$A_1 \times A_1$ $\quad \Phi = \{\pm\alpha, \pm\beta\}$ $\quad (\beta,\alpha)=0$

$\sigma_\alpha, \sigma_\beta$

$\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$

$W = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

A_2



$W = \langle \sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta} \rangle$

$\sigma_\alpha(\beta) = \alpha + \beta$

$\Rightarrow \sigma_\alpha \sigma_\beta \sigma_\alpha = \sigma_{\sigma_\alpha(\beta)} = \sigma_{\alpha+\beta}$

$$\Rightarrow W = \langle \sigma_\alpha , \sigma_\beta \rangle$$

$$\sigma_\alpha (\alpha, \beta) = (\alpha \;\; \beta) \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_\beta \qquad\qquad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\sigma_\alpha \sigma_\beta \sigma_\alpha \cdots\cdots$$

$$\sigma_\alpha \sigma_\beta = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

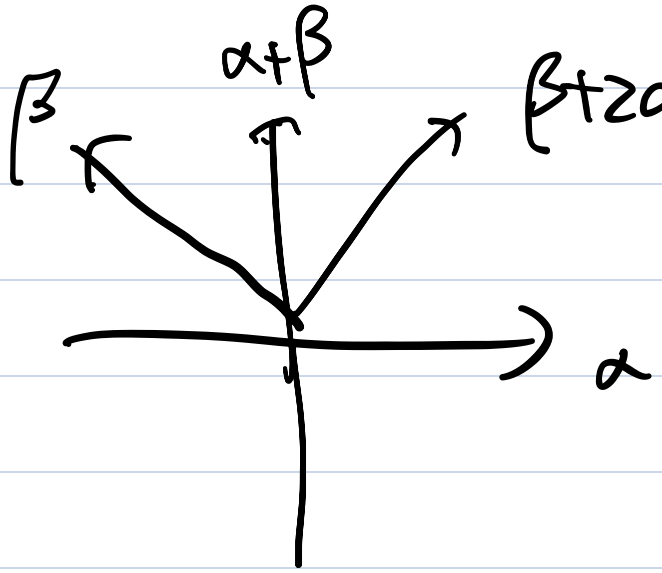
$$(\sigma_\alpha \sigma_\beta)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$W = D_3$

$\sigma_\alpha \quad \sigma_\beta \qquad \sigma_\alpha\sigma_\beta\sigma_\alpha$

$\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \qquad \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$, Id

B_2



$W = \langle \sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta}, \sigma_{\alpha+2\beta} \rangle$

$\sigma_\beta(\alpha) = -\alpha-\beta$

$$\sigma_{\alpha+\beta} = \sigma_{\beta}\sigma_{\alpha}\sigma_{\beta}$$

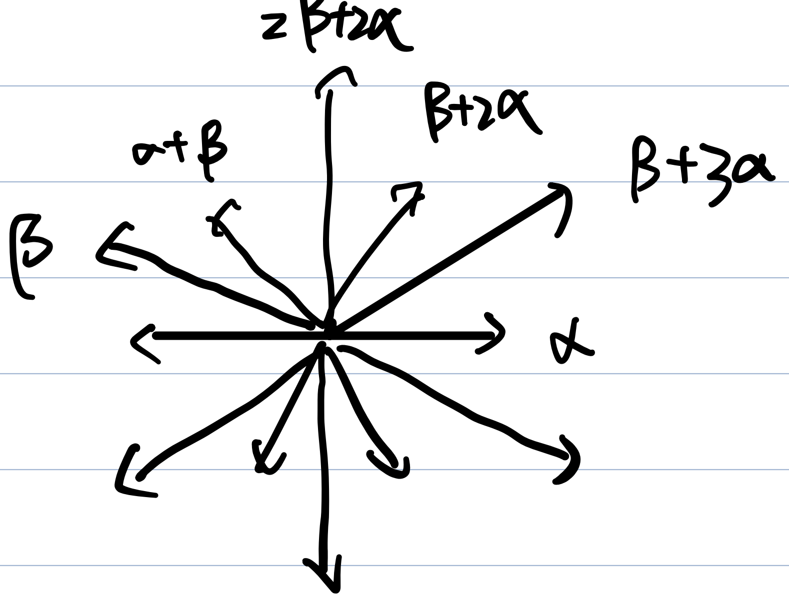
$$\sigma_{\beta+2\alpha} = \sigma_{\alpha}\sigma_{\beta}\sigma_{\alpha}$$

$$W = \langle \sigma_{\alpha}, \sigma_{\beta} \rangle$$

$$\sigma_{\alpha} \sim \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_{\beta} \sim \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

$W \Rightarrow D_4$. 8 elements,



$$\sigma_{\alpha+\beta} = \sigma_\beta \sigma_\alpha \sigma_\beta$$

$$\sigma_\alpha(\beta) = -\beta - 3\alpha$$

$$\sigma_\beta(\beta+3\alpha) = -2\beta - 2\alpha$$

$$W = \langle \sigma_\alpha, \sigma_\beta \rangle = D_4.$$

§10. Simple Roots and Weyl group

(E, Φ) root system.

§10.1. Base and Weyl chambers

Definition 10.1.

$\Delta \subseteq \Phi$ is called a basis of Φ

(B1) Δ is a basis of Φ

(B2) $\forall \beta \in \Phi$, $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, then

① $k_\alpha \in \mathbb{Z}_{\geq 0}$

or ② $k_\alpha \in \mathbb{Z}_{\leq 0}$, $\forall \alpha \in \Delta$

The roots in Δ are called simple.

Def. Δ is a basis of Φ

(1) the height of $\beta \in \Phi$,

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

$$ht(\beta) := \sum_{\alpha \in \Delta} k_\alpha \quad (\text{relative to } \Delta).$$

(2) If $k_\alpha \geq 0$, $\forall \alpha$, call β a

positive root $\quad \beta > 0$

(3) ≤ 0

negative root. $\quad \beta < 0$

(4) $\Phi^+ = \{\beta \in \Phi \mid \mathrm{ht}\,\beta > 0\}$

$\Phi^- =$ < 0

(5). Partial order on E.

$\gamma_1, \gamma_2 \in E$.

$\gamma_1 < \gamma_2 \Longleftrightarrow \gamma_2 - \gamma_1 = \sum k_\alpha \alpha$

$k_\alpha \in \mathbb{Z}_{\geq 0}$

Lemma 10.3.

Δ is a base of Φ

If $\alpha \neq \beta \in \Delta$. then $(\alpha, \beta) \leq 0$

and $\alpha-\beta,\ \beta-\alpha \notin \Phi$.

Theorem. 10-4.

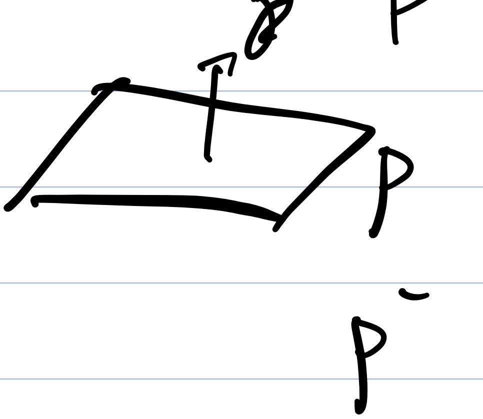
Φ has a base.

Construction: $\Phi \subseteq E$.

$\gamma \in E$, define

$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\alpha, \gamma) > 0\}$

$\Phi \cap P^+$



$\gamma \in E$

(1) γ is regular, iff $\gamma \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$,

Otherwise γ is singular

* $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha \neq \phi$. $\Rightarrow$ Exists regular elements.

(2) γ is regular

$$\Leftrightarrow \Phi^+(\gamma) \sqcup \overset{\Phi^+(\gamma)}{\overset{\shortparallel}{\Phi^-(\gamma)}} = \Phi.$$

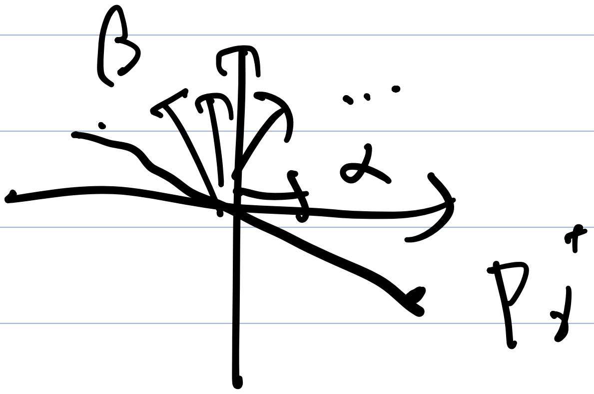
(3) α is called decomposible if

$\alpha = \beta_1 + \beta_2$, $\beta_i \in \Phi^+(\gamma)$

Otherwise, indecomposible.

For example.

rank = 2



α, β indecom.

Theorem. 10.5.

Let γ regular, then

(1) $\Delta(\gamma) = \{$ indecom roots in $\Phi^+(\gamma) \}$

is a base of Φ

(2) Δ is a base of Φ

$\Rightarrow \exists\ \gamma$ regular s.t.

$\Delta = \Delta(\gamma)$

Pf:

① $\forall \alpha \in \Phi^+(\gamma)$ is a $\mathbb{Z}_{\geq 0}$ linear combination of $\Delta(\gamma)$

Otherwise, $|\Phi^+(\gamma)| < +\infty$

Let $\beta \in \Phi^+(\gamma)$ be such that with (β, γ) minimal.

β is decom. by the assumption,

$\Rightarrow \quad \beta = \beta_1 + \beta_2 \ , \quad \beta_i \in \Phi^+(\gamma)$

$0 < (\beta_i, \gamma) < (\beta, \gamma)$, $\quad \times$.

② if $\alpha \neq \beta \in \Delta(\gamma)$, then $(\alpha, \beta) \leq 0$

otherwise $(\alpha, \beta) > 0$

$\Rightarrow \quad \alpha - \beta, \ \beta - \alpha \in \Phi$

$\alpha - \beta \in \Phi^+(\gamma)$ or $\beta - \alpha \in \Phi^+(\gamma)$

$\Rightarrow$ One of α, β is decom-

③ $\Delta(\gamma)$ is linearly indep.

Fact: $M \subseteq E$ if $\underline{\exists r} \in E$ s.t.

$(r, \alpha) > 0$, $\forall \alpha \in M$

and $\forall \alpha \neq \beta \in M$, $(\alpha, \beta) \leq 0$

$\Rightarrow$ M is linearly indep.

pf: Assume $\sum_{\alpha \in M} k_\alpha \alpha = 0$

$I_1 = \{ \alpha \in M \mid k_\alpha \geq 0 \}$

$I_2 = \{ \alpha \in M \mid k_\alpha < 0 \}$

$$\Rightarrow \sum_{\underset{\geq 0}{\alpha \in I_1}} k_\alpha \alpha = - \sum_{\beta \in I_2} \underset{<0}{k_\beta} \beta = \theta$$

$$0 \leq (\theta, \theta) = \sum_{\alpha \in I_1} \sum_{\beta \in I_2} \underbrace{-k_\alpha k_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0}$$

≤ 0

$\Rightarrow \quad \theta = 0.$

$$0 = (\theta, \gamma) = \sum_{\alpha \in I_1} \underbrace{k_\alpha}_{>0} \underbrace{(\beta, \gamma)}_{>0}$$

$$\Rightarrow k_\alpha = 0, \ \forall \alpha \in I_1, I_2$$

$\Rightarrow \Delta(\gamma)$ is a base (1) ✓.

(2) $\Delta = \{\alpha_1, \cdots, \alpha_l\}$

$G = ((\alpha_i, \alpha_j))$ non-deg of

$(,) \Rightarrow G$ invertible.

$\exists \ G_1, \cdots, G_k$ s.t. $(a_1, \cdots, a_l)G$

$$\shortparallel$$

$$(1, \cdots, 1)$$

$$r \triangleq \sum_{i=1}^{l} a_i \alpha_i$$

$$\Rightarrow (\gamma, \alpha_i) = 1 \quad , \forall i$$

$$\Rightarrow (\gamma, \beta) \neq 0, \quad \forall \beta \in \Phi$$

$\Rightarrow \gamma$ regular

$$\Phi^+(\gamma) = \Phi^+. \quad \underline{\Delta(\gamma) = \Delta}.$$

$$\square$$

Def. 10.6.

$E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$

$\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha \iff \gamma$ regular.

(1) The connected components of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ is called the open Weyl chambers of E

(2) γ regular $\Rightarrow \gamma \in$ Unique Wely chamber, denotes by $C(\gamma)$

(3) $C(\gamma) = C(\gamma')$

$\iff \gamma, \gamma'$ on the same side of

P_α for $\forall$ $\alpha \in \Phi$.

$\Leftrightarrow \quad \Phi^+(\gamma) = \Phi^+(\gamma')$

$\Leftrightarrow \quad \Delta(\gamma) = \Delta(\gamma')$

(4). if $\Delta = \Delta(\gamma)$

$C(\Delta) \triangleq C(\gamma)$

is called the fundamental Weyl chamber relative to Δ.

(5) $C(\Delta) = \{ \gamma \in E \mid (\gamma, \alpha) > 0, \forall \alpha \in \Delta \}$

Facts: $W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$

(1) If $\sigma \in W$, then $\sigma(\Delta)$ is a base of Φ

(2) $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$, $\underline{\sigma \in W}$

Pf: $\sigma(\gamma)$ is regular,

Otherwise $\exists \alpha \in \Phi$, $(\sigma(\gamma), \alpha) = 0$

$\Leftrightarrow (\gamma, \sigma^{-1}(\alpha)) = 0$ ✓.

$(\sigma(\alpha), \sigma(\gamma)) = (\alpha, \gamma)$

$\Rightarrow \forall \alpha \in A(\gamma)$

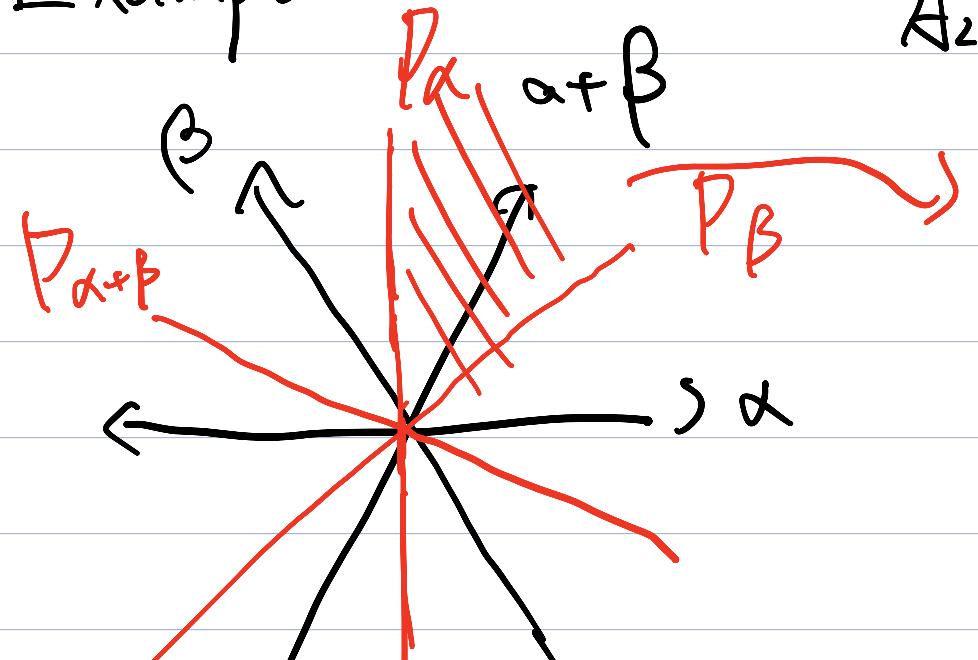
$\sigma(\Phi^+(\gamma)) > \Phi^+(\sigma(\alpha))$

$\forall \alpha \in \sigma(\Delta\gamma))$ is indp in

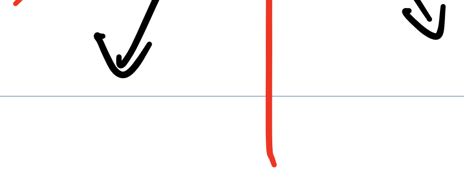
$\Phi^+(\sigma(\alpha))$

$\Rightarrow \sigma(C(\gamma)) = C(\sigma(\gamma))$

Example. A_2.



fundamental chamber.



$\#$ Weyl chambers $= |W| = |S_3|$.

§10.2. Lemmas on simple roots.

Lemma 10.7.

If $\alpha \in \Phi^+ \setminus \Delta$

then $\exists\ \beta \in \Delta$ s.t. $\alpha - \beta \in \Phi^+$

Proof: Otherwise, $\forall\ \beta \in \Delta$,

$(\alpha, \beta) \le 0$ (Lemma 9.8.

$\alpha = \sum k_i \beta_i$

$\Rightarrow (\alpha,\alpha) = \sum k_i (\alpha, \beta_i) \leq 0$, $\times$.

$ht(\alpha) = 1 + ht(\alpha - \beta)$

Corollary 10.8.

$\forall \beta \in \Phi^+$, $\exists\, \alpha_i \in \Delta$, $i = 1, \dots, k$

$k = ht(\beta)$ s.t.

$$\beta = \alpha_1 + \dots + \alpha_k \ , \ \alpha_i \in \Delta .$$

pf: ✓.

Lemma 10.9. $\alpha \in \Delta$, then

$$\sigma_\alpha \in S_{\Phi^+ \setminus \{\alpha\}}$$

pf: $\forall \beta \in \Phi^+ \setminus \{\alpha\}$

$$\beta = \sum_{\theta \in \Delta \setminus \{\alpha\}} k_\theta \theta + k_\alpha \alpha$$

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

$$= * + \underset{>0}{\underline{k_\theta}}\, \theta + \cdots$$

$\Rightarrow \sigma_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$

Corollary 10.10.

Set $\delta = \frac{1}{2}\sum_{\beta\in\Phi^+} \beta$

Then $\sigma_\alpha(\delta) = \delta - \alpha$, $\forall \alpha \in A$

Pf: $\forall \alpha \in \Delta$

$$\sigma_\alpha(\delta) = \sigma_\alpha\left(\frac{1}{2}\sum_{\beta\neq\alpha}\beta + \frac{1}{2}\alpha\right)$$

$$= \frac{1}{2}\sum_{\tau\neq\alpha}\tau - \frac{1}{2}\alpha$$

$$= \delta - \alpha$$

$\Rightarrow A(\delta) = A$.

Lemma 10.11.

$\alpha_1 \cdots \alpha_t \in \Delta$

$\sigma_i \triangleq \sigma_{\alpha_i}$

If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t) \in \Phi^-$

then $\exists\ 1 \leq s < t$ s.t.

$$\sigma_1 \sigma_2 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$$

pf: Write $\beta_i \triangleq \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t) \in \Phi$, $0 \leq i \leq t-1$

$\exists\ 1 \leq s \leq t-1$, $\beta_s \in \Phi^+$, $\beta_{s-1} \in \Phi^-$

$\beta_{s-1} = \sigma_s(\beta_s) \quad \beta_s \in \Phi^+$

Lemma 10.9 $\Rightarrow \beta_s = \alpha_s$.

$\Rightarrow \alpha_s = \underbrace{\sigma_{s+1} \cdots \sigma_{t-1}}_{\sigma}(\alpha_t)$

$= \sigma(\alpha_t)$

$\Rightarrow \sigma_{\alpha_s} = \sigma_{\sigma(\alpha_t)} = \sigma \sigma_{\alpha_t} \sigma^{-1}$

$= \sigma_{s+1} \cdots \sigma_{t-1} \underrightarrow{\boxed{\sigma_t \sigma_{t-1} \cdots \sigma_{s+1}}}$

$\Rightarrow \sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$

Corollary. 10.12

If $\sigma = \sigma_1 \cdots \sigma_t$. $\sigma_i \overset{\Delta}{=} \sigma_{\alpha_i}$, $\alpha_i \in \Delta$ with t as small as possible $\Rightarrow \sigma(\alpha_t) \in \Phi^-$

that is, σ can't be expressed as $\sigma = \sigma'_1 \cdots \sigma'_s$, $s<t$.

Pf: Otherwise

$$\begin{array}{c} \sigma(\alpha_t) \in \Phi^+ \\ \parallel \\ -\sigma_1 \cdots \sigma_{t-1}(\alpha_t) \end{array}$$

$\Leftrightarrow \quad \sigma_1 \cdots \sigma_{t-1}(\alpha_t) \in \Phi^-$, X.

Theorem 10.13. $\Delta \subseteq \Phi$, W

$W' \overset{\Delta}{=} \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$ is a subgroup of W

(a) If γ is regular, then $\exists\, \sigma \in W'$ s.t. $(\sigma(\gamma), \alpha) > 0$, $\forall \alpha \in \Delta$

(W' acts transitively on Chambers).

Pf: $\{\sigma(\gamma) \mid \sigma \in W\}$ is a finite set

Choose $\sigma \in W'$ s.t. $(\sigma(\gamma), \underset{\frac{1}{2}\sum_{\beta\in\Phi^+}\beta}{\underset{\shortparallel}{\delta}})$ is

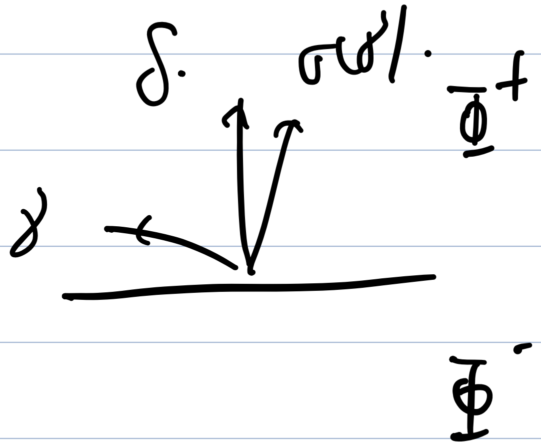
Largest

$\Rightarrow \forall \alpha \in A$, $\sigma_\alpha \sigma \in W'$

$\Rightarrow (\sigma_\alpha \sigma(\gamma), \delta) \leq (\sigma(\gamma), \delta)$

$\Leftrightarrow (\sigma(\gamma), \underset{\delta - \alpha.}{\underset{\shortparallel}{\sigma_\alpha(\delta)}}) \leq (\sigma(\gamma), \delta)$

$\Leftrightarrow (\sigma(\gamma), \alpha) \geqslant 0.$



(b) If Δ' is a base of Φ,

then $\exists\ \sigma \in W'$ s.t. $\sigma(\Delta') = \Delta$

Pf: Δ' is a base

$\Rightarrow \exists\ \gamma,\ \Delta' = \Delta(\gamma)$

By (a), $\exists\ \sigma \in W' \subseteq W,\ \sigma(\gamma) \in \mathfrak{C}(\Delta)$

$\Rightarrow \sigma(\Delta') = \sigma(\Delta(\gamma))$

$= \Delta(\sigma(\gamma)) = \Delta$

$\square$

(c). If $\alpha \in \Phi$, then $\exists\ \sigma \in W'$ s.t.

$\sigma(\alpha) \in \Delta$

Pf: $P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$ is non empty.

Take $\gamma \in P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$

$|\Phi| < +\infty$

Let $\varepsilon_1 > 0$ s.t. $|(\gamma, \beta)| \geq \varepsilon_1$,

$\forall \beta \in \Phi \setminus \{\pm\alpha\}$

Choose $t > 0$ s.t.

$|t(\alpha, \beta)| \leq \varepsilon_1$, $\forall \beta \in \Phi \setminus \{\pm\alpha\}$

and $t(\alpha, \alpha) = \varepsilon < \varepsilon_1$

For $\gamma' := \gamma + t\alpha$

$(\gamma', \alpha) = t(\alpha, \alpha) = \varepsilon$

$|(\gamma', \beta)| = |(\gamma', \beta) + t(\alpha, \beta)|$

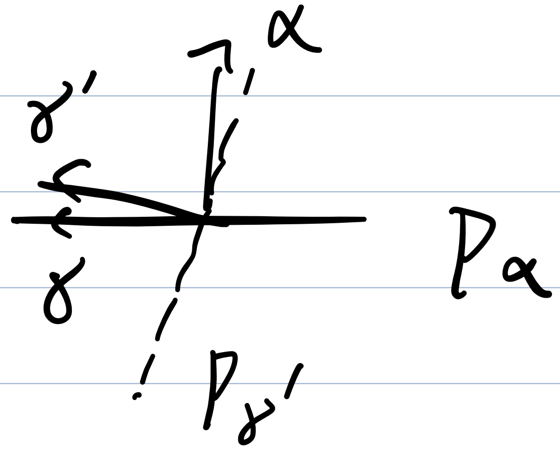
$\geq \big| |(\gamma, \beta)| - |t(\alpha, \beta)| \big|$

$> \varepsilon_1 > \varepsilon$

If $\alpha = \beta_1 + \beta_2$

$\beta_i \in \Phi^+(\gamma') \Rightarrow (\gamma', \beta_i) > 0$

$\Rightarrow (\gamma', \beta_i) > \varepsilon_1$



$\Rightarrow (\gamma', \alpha) > 2\varepsilon$, Contradiction!

(d). $W = W' = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$

Pf: (a)–(c) hold for W'.

$\forall \beta \in \Phi$. By (c)

$\exists \sigma \in W'$ s.t. $\sigma(\beta) = \alpha \in \Delta$

$\Rightarrow \beta = \sigma^{-1}(\alpha)$

$\sigma_\beta = \sigma_{\sigma^{-1}(\alpha)} = \sigma^{-1} \sigma_\alpha \sigma \in W'$

$\Rightarrow w \in W'$

1e). $\sigma \in W$

If $\sigma(\Delta) = \Delta$

$\Leftrightarrow \sigma = Id_E$

($W \curvearrowright$ chambers transitively & fai~~th~~fully).

$\overset{1:1}{\longleftrightarrow}$

Pf: $\sigma(\Delta) = \Delta$

$|W| = \#$ chambers.

$\Rightarrow \sigma(\Phi^+) = \Phi^+$

$\Rightarrow \sigma(\Phi^+) = \Phi^+$

By Cor 10.12 $\sigma = \mathrm{Id}_E$

$\square$

$\{\sigma(\alpha) \mid \sigma \in W, \alpha \in \Delta\} = \Phi$

$|\Phi| \leq |W| \dim E$.

Definition 10.14.

(1) $\forall \sigma \in W$ $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$

$\alpha_i \in \Delta$, t minimal

We call the expression reduced

$t = l(\Delta)$, the length of σ

relative to Δ

If $l(\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}) = t$, $\alpha_i \in \Delta$

$\Rightarrow$ $l(\sigma_{\alpha_i} \cdots \sigma_{\alpha_j}) = j-i+1$

(2) $n(\sigma) \triangleq |\sigma(\Phi^+) \cap \Phi^-|$

By Lemma 10.9. $\forall \alpha \in \Delta$

$n(\sigma_\alpha) = 1$

$n(\sigma) = 0 \Rightarrow \sigma(\Delta) = \Delta \Rightarrow \sigma = Id_E$

Lemma 10.15.

$\forall \sigma \in W, \quad n(\sigma) = \ell(\sigma)$

Pf: Induction on $\ell(\sigma)$

(1) $\ell(\sigma) = 0 \Leftrightarrow \sigma = Id \Leftrightarrow n(\sigma) = 0.$

$\ell(\sigma) = 1 \Leftrightarrow \sigma = \sigma_\alpha \Rightarrow n(\sigma) = 1$

(2) Assume $\forall \tau \in W, \ \ell(\tau) < \ell(\sigma)$

then $n(\tau) = \ell(\tau)$

$\ell(\sigma) = t \qquad \sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$

By Cor 10.12.

$\sigma(\alpha_t) \in \Phi^-$

$$\ell(\sigma\sigma_t) = t+ = n(\sigma\sigma_t)$$

Find $n(\sigma)$.

① $\sigma\sigma_\alpha(\alpha) = \sigma(-\alpha) \in \Phi^+$

$\sigma\sigma_\alpha(\Phi^+) \cap \Phi^-$

$= \sigma(\sigma_\alpha(\Phi^+ \setminus \{\alpha\}) \cap \Phi^-)$

$= \sigma(\Phi^+ \setminus \{\alpha\}) \cap \Phi^-$

$= (\sigma(\Phi^+) \cap \Phi^-) \setminus \{\sigma(\alpha)\}$

$\Rightarrow\ t-1 = n(\sigma) - 1$

$\forall$ regular γ $\exists\, \sigma \in W$, $\sigma(\gamma) \in C(\Delta)$

$\overline{C(\Delta)} \triangleq \{ \gamma \in E \mid (\gamma, \alpha) \geq 0, \ \forall \alpha \in \Delta \}$

is a fundamental domain for the action of W on E.

Lemma 10.16.

$\lambda, \mu \in \overline{C(\Delta)}$

If $\sigma\lambda = \mu$, then $\lambda = \mu$

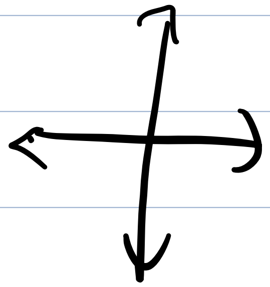
§10.4. Irre. root systems.

Def. 10.17.

Φ is call reducible if $\Phi = \Phi_1 \cup \Phi_2$, $\Phi_i \neq \phi$, $(\Phi_1, \Phi_2) = 0$

Otherwise, irreducible.

* $A_1 \times A_1$ reducible



* A_2, B_2, G_2, A_1 irre.

Prop. 10.18

$\Delta \subseteq \Phi$ base

$\Rightarrow$ Φ is reducible

$\Leftarrow$ Δ is "reducible"

$(\Delta = \Delta_1 \cup \Delta_2, \quad \Delta_i \neq \phi, \ (\Delta_1, \Delta_2) = 0)$.

Pf: $(\Rightarrow)$:

$\Delta_i = \Delta \cap \Phi_i$

$(\Delta_1, \Delta_2) = 0, \quad \Delta = \Delta_1 \cup \Delta_2$

If $\Delta_1 = \phi \Rightarrow \Delta \subseteq \Phi_2$

$\Rightarrow E = \mathrm{Span}(\Phi_2)$ $\quad\quad$ $\times$.

(⇐): Assume $\Delta = \Delta_1 \cup \Delta_2$

$(\alpha, \beta) = 0 \Rightarrow \sigma_\alpha(\beta) = \beta$

$\Rightarrow \sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$

Set

$\Phi_i = W(\Delta_i) = \{\sigma(\alpha) \mid \alpha \in \Delta_i, \sigma \in W\}$

Claim: Thm 10.3 $\Rightarrow \Phi_1 \cup \Phi_2 = \Phi$

$\Phi_i \subseteq \mathrm{Span}_{\mathbb{R}} \Delta_i$

$\forall \alpha \in \Delta_1, \beta \in \mathrm{Span}_{\mathbb{R}} \Delta_2$

$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta$

$W = \langle \sigma_\alpha, \sigma_\beta \mid \alpha \in \Delta_1, \beta \in \Delta_2 \rangle$

$\Rightarrow W(\Delta_i) \subseteq \mathrm{Span}_{\mathbb{R}} \Delta_i$

$\Rightarrow \Phi$ irre $\Leftarrow$ Δ irre.

Lemma 10.19

Φ irr. Relative to the

partial order ($\alpha < \beta \Leftrightarrow \beta - \alpha \in \Phi^+$).

$\Rightarrow$ There is a unique maximal root θ, Moreover

(1) $\gamma \neq \theta$, $\forall \in \Phi$

$\Rightarrow ht(\gamma) < ht(\theta)$, $(\theta, \alpha) \geq 0$, $\forall \alpha \in \Delta$

(2) If $\theta = \sum_{\alpha \in A} k_\alpha \alpha$

$\Rightarrow k_\alpha > 0$, $\forall \alpha \in \Delta$

Pf: Let $\theta = \sum k_\alpha \alpha$ maximal

$$_{\alpha\in\Delta}$$

(By finiteness).

then $\theta \in \Phi^+ \Rightarrow k_\alpha \geq 0$

$$\Delta_1 = \{\alpha \in \Delta \mid k_\alpha > 0\}$$

$$\Delta_2 = \{\alpha \in \Delta \mid k_\alpha = 0\}$$

$$\Rightarrow \theta = \sum_{\alpha\in\Delta_1} k_\alpha \alpha$$

By Lemma 10.3.

$$\alpha \neq \beta \in \Delta \Rightarrow (\alpha, \beta) \leq 0$$

$$\Rightarrow \forall \alpha \in \Delta_1, \quad \beta \in \Delta_2$$

$(\alpha, \beta) \leq 0$

$(\beta, \theta) \leq 0$

If $\exists \beta \in \Delta_2$, $(\beta, \theta) < 0$

$\Rightarrow \theta + \beta \in \Phi$, $\times$.

$\Rightarrow \forall \beta \in \Delta_2$, $\forall \alpha \in \Delta_1$

$(\alpha, \beta) = 0$

$\Rightarrow \Delta = \Delta_1 \cup \Delta_2$

Δ irre. $\Rightarrow \Delta_2 = \phi$

Pf of (r):

It suffices to prove the maximal element is unique.

If θ' is another maximal element, $\theta' = \sum_{\alpha\in\Delta} k'_\alpha \alpha$

$k'_\alpha > 0$

$(\theta, \theta') = \sum_{\alpha\in\Delta} k'_\alpha (\theta, \alpha)$

$k'_\alpha > 0$, $(\theta, \alpha) \geq 0$ (or $\theta+\alpha \in \Phi$).

$\exists \alpha, \quad (\theta, \alpha) > 0$

$\Rightarrow (\theta, \theta') > 0$

$\theta = \theta'$ ✓. or $\underline{\theta - \theta' \in \Phi}$

$\theta - \theta' \notin (\Phi^+ \cup \Phi^-) = \Phi$

$\Rightarrow \theta = \theta'$

Lemma 10.20 Φ irre.

$\Rightarrow$

* (1) W acts irre. on E

(group representation)

(2) $\forall \alpha \in \Phi$

$\text{Span}\, W(\alpha) = E$.

Pf: $\text{Span}_{\mathbb{R}} W(\alpha)$ is W invariant

(1) $\Rightarrow$ (2).

Let $0 \neq E' \subseteq E$ is W-invariant

Let $E'' = \{ x \in E \mid (x, E') = 0 \}$

$\Rightarrow$ $E = E' \oplus E''$

Let $\Phi_1 = E' \cap \Phi$

$\Phi_2 = E'' \cap \Phi$

$(\Phi_1, \Phi_2) = 0$

Claim: E'' is W-invariant,

$\Phi_1 \cup \Phi_2 = \Phi$

① $(\alpha, \sigma(\beta)) = (\sigma^{-1}(\alpha), \beta))$

$\forall \sigma \in W$

② If $\alpha \notin E'$, $\alpha \in \Phi$

Since $E = \mathbb{F}\alpha \oplus P_\alpha$

$\forall\ x \in E'$, $x = a\alpha + \beta$

$a \in \mathbb{F} \quad \beta \in P_\alpha$

$\sigma_\alpha(x) \in E'$

$$\Rightarrow \quad \underset{\displaystyle 2a\alpha}{\underset{\shortparallel}{x - \sigma_\alpha(x)}} \in E'$$

$\Rightarrow \quad a = 0$

$\Rightarrow \quad E' \subseteq P_\alpha$

$\Rightarrow \alpha \in (E')^{\perp} = E''$

$\Rightarrow \Phi = \Phi_1 \cup \Phi_2 \quad (\Phi_1, \Phi_2) = 0$

$\Rightarrow \Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$ ✓.

$\Rightarrow E' = E$

Lemma 10.2 | Φ irre.

① $|\{\|\alpha\| \mid \alpha \in \Phi\}| \leq 2$

② If $\|\alpha\| = \|\beta\|$

$\Rightarrow \exists\, \sigma \in W \quad \sigma(\alpha) = \beta$

Pf: (1) By $\underline{\text{Lemma 10.20}}$

$\forall\, \alpha, \beta \in \Phi$

$\exists\, \sigma \in W \quad \underline{(\sigma(\beta), \alpha) \neq 0}$

WLOG: $(\alpha, \beta) \neq 0$

$\Rightarrow \|\alpha\|^2 / \|\beta\|^2 = 1, 2, 3, 1/2, 1/3$

If $\underbrace{\|\alpha\| \underbrace{>}_{2/3} \|\beta\| \underbrace{>}_{2/3} \|\gamma\|}_{}$ $\times$.

2/3

② $\|\alpha\| = \|\beta\|$

$\Rightarrow \exists \sigma \in W, \ (\sigma(\beta), \alpha) > 0$

$\|\sigma(\beta)\| = \|\alpha\|$

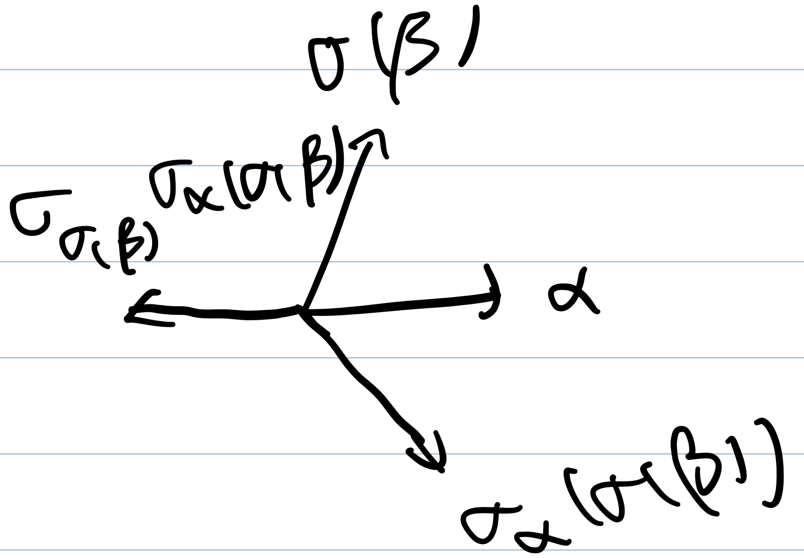
$\Rightarrow \ <\sigma(\beta), \alpha> = 1$

$\sigma_\alpha(\sigma(\beta)) = \sigma(\beta) - <\sigma(\beta), \alpha>\alpha$

$= \sigma(\beta) - \alpha$

$\sigma_{\sigma(\beta)}(\alpha) = \alpha - \sigma(\beta)$

$$\sigma_\alpha \sigma_{\sigma(\beta)} \sigma_\alpha (\sigma(\beta)) = \alpha$$



Def 10.22

Φ irre. with two distinct root lengths, Long roots/short roots.

If all equal $\Rightarrow$ long roots.

Lemma 10.23.

Φ irr. with 2 distinct root lengths, then the maximal root θ is long

Pf: $\forall \alpha \in \Phi, \exists \sigma, \sigma(\alpha) \in \Delta$

$\theta + \alpha \notin \Phi, \forall \alpha \in \Delta \Rightarrow (\theta, \alpha) \geq 0$

$\Rightarrow \theta \in \overline{C(\Delta)}$

$\theta - \alpha > 0$

$\forall \gamma \in C(\Delta)$

$$(\gamma, \theta - \alpha) \geq 0$$

$$\Rightarrow \theta \in \overline{C(A)}$$

$$\alpha \in \mathbb{E}, \ \exists \sigma \in W$$

$$\sigma(\alpha) \in \overline{C(A)}, \quad (\sigma(\alpha), \delta)$$

maximal

$$(\theta, \theta - \sigma(\alpha)) \geq 0$$

$$\Rightarrow (\theta, \theta) \geq (\theta, \sigma(\alpha)) \geq (\alpha, \alpha)$$

D

§11. classification.

§11.1. Cartan matrix of Δ

Def. 11.1.

$\Delta = \{\alpha_1, \cdots, \alpha_l\}$

$l = \dim E$.

$C \triangleq (\langle \alpha_i, \alpha_j \rangle)_{ll} \in M_l(\mathbb{Z})$

C is called the Cartan

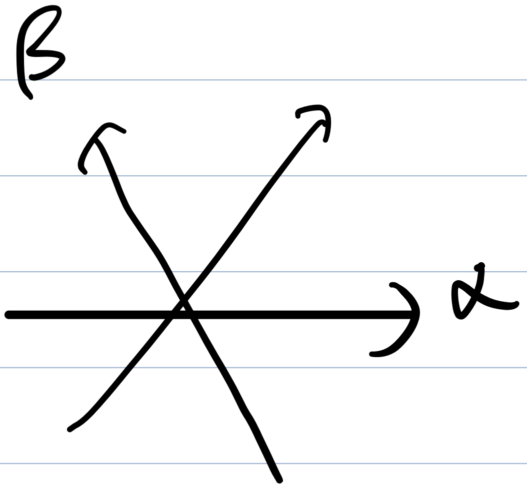
matrix. of Φ

Example. 11.2

A_1 (2)

$A_1 \times A_1$ $\{\pm\alpha, \pm\beta\}$ $C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

A_2

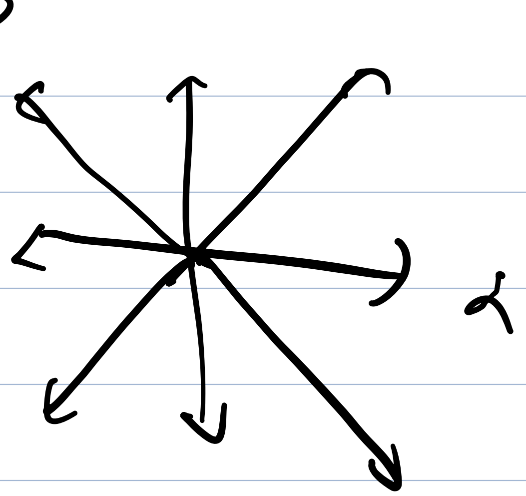


$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

B_2

$C = \begin{pmatrix} 2 & -1 \end{pmatrix}$

$-2 \quad 2)$



G_2

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Facts.

(1) C depends on the ordering of simple roots

$\{\beta_1, \cdots, \beta_l\} = \{\alpha_1, \cdots, \alpha_l\}$

S permutation

$(\beta_1 \cdots \beta_l) = (\alpha_1 \cdots \alpha_l) S$

$C' = SCS^T$

(2) C is indep of the choice of Δ

Δ , Δ'

$\exists\ \sigma \in W \qquad \sigma(\Delta) = \Delta'$

$\Rightarrow C = C'$.

(3) C is nonsingular ($\det C \neq 0$).

$$C = \left(\langle \alpha_i, \alpha_j \rangle \right)$$

$$= \left((\alpha_i, \alpha_j) \right) \operatorname{diag}\left(\frac{2}{(\alpha_j, \alpha_j)} \right)$$

$\det C > 0.$ ✓.

$$\Phi \longrightarrow \Delta \longrightarrow C$$

Prop 11.3. The Cartan matrix determines Φ up to isomorphism.

$\Phi' \subseteq E'$ root system

$\Delta' = \{ \alpha_1', \cdots, \alpha_l' \}$ base

If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle$

$\alpha_i \to \alpha_i'$ extends uniquely to

an isomorphism $\varphi : E \to E'$

mapping Φ onto Φ and satisfying

$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$

proof: $\sigma_{\alpha_i'}(\alpha_j') = \sigma_{\varphi(\alpha_i)}(\varphi(\alpha_j))$

$= \sigma_{\alpha_i'}\varphi(\alpha_j)$

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[d, "u'"] \\
E \arrow[r, "\varphi"] & E'
\end{tikzcd}$$

$$u \xrightarrow{\sim} u'$$

$$\sigma \longrightarrow \varphi \circ \sigma \circ \varphi^{-1}$$

$\forall \beta \in \Phi$ $\qquad\qquad\qquad\qquad$ $\alpha \in \Delta$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$$\varphi(\beta) = \varphi \circ \sigma(\alpha) = \varphi \circ \sigma \circ \varphi^{-1}(\varphi(\alpha))$$

$$\in \Phi'$$

$\Rightarrow \quad \varphi(\Phi) \subseteq \Phi'$

$$\varphi^{-1}(\Phi') \subseteq \Phi$$

$$\Rightarrow \qquad \varphi(\Phi) = \Phi'$$

Remark 11.4.

① $\Delta, C \rightarrow$ recover Φ

② $\alpha, \beta \in \Delta \quad \beta \neq \alpha$

$$\{\beta \pm i\alpha \mid \underset{\overset{\uparrow}{0}}{-r} \leq i \leq q\} \qquad r - q = \langle \beta, \alpha \rangle$$

Example 11.5. G_2.

$\Delta = \{\alpha, \beta\}$ $\quad$ $C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$\|\beta\| > \|\alpha\|$

(1) $ht = 1$ $\quad$ α, β $\quad$ $\langle \beta \alpha \rangle = -3$

(2) $ht = 2$ $\quad$ $\beta + \alpha$

$\alpha + \beta$

(3) $\beta + 2\alpha$ $\quad$ $\underline{\boxed{\alpha + 2\beta \notin \Phi}}$

(4) $\beta + 3\alpha$ [must come from (3)].

(5) $\beta + 4\alpha \notin \Phi$.

$$\langle \beta+3\alpha, \beta \rangle = -1$$

$$\{\beta+3\alpha+i\beta \mid -\overset{0}{\underset{\sim}{r}} \leq i \leq q\}$$

$$r-q=-1.$$

$$\Rightarrow q=1$$

$$\Rightarrow 2\beta+3\alpha \in \Phi.$$

(6) $2\beta+3\alpha+\beta \notin \Phi$

$$2\beta+3\alpha+\alpha \notin \Phi$$

$$\Phi = \Phi^{+} \cup (-\Phi^{+}).$$

$* \ \Phi = \{\sigma(\alpha) \mid \alpha \in \Delta, \sigma \in W\}$ $\quad G_2$.

$\Delta = \{\alpha, \beta\} \qquad C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$$\sigma_\alpha(\alpha, \beta) = (\alpha, \beta)\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_\beta(\alpha, \beta) = (\alpha, \beta)\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

find $\langle \sigma_\alpha, \sigma_\beta \rangle (\alpha, \beta)$

---

§11.2. Coxeter graphs and Dinkin diagrams

Δ, $C = (\langle \alpha_i, \alpha_j \rangle)$

$= (\alpha_{ij}).$

$\alpha_{ij} = 0 \Leftrightarrow \alpha_{ji} = 0.$

Def. 11.6

The Coxeter graph of Φ

having ℓ vertices

ith to jth by

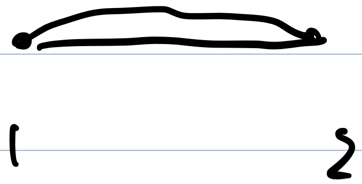
$\underline{\{0,1,2,3\}} \ni \boxed{\underline{\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle}}$ edges. $i \neq j$.

A_1 :

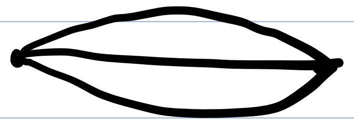
$A_1 \times A_1$:

reducible

B_2 :



G_2 :



Remark :

(1) Coxeter graph $\nrightarrow$ Cartan matrix

(2) Coxeter graph $\longrightarrow$ Weyl group

(proof is hard).

Claim: Φ, Δ, ,

$$W = \langle \sigma_{\alpha_i} \mid \sigma_{\alpha_i}^2 = 1 = (\sigma_{\alpha_i} \sigma_{\alpha_j})^{m_{ij}} \rangle$$

$$1 \le i \le 6$$

$\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$	m_{ij}
0	2
1	3
2	4
3	6

A_2. $W = \langle \sigma_1, \sigma_2 \mid \sigma_i^2 = 1 = (\sigma_1 \sigma_2)^3 \rangle$

$= S_3$

Claim 2. $\alpha, \beta \in \Phi$ the order of $\sigma_\alpha \sigma_\beta$ depends on $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$

pf 1: check for all $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$

pf 2: $E = \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus (P_\alpha \cap P_\beta)$

$$\sigma_\alpha \sigma_\beta \Big|_{P_\alpha \cap P_\beta} = Id$$

$\Rightarrow$ the order of $\sigma_\alpha \sigma_\beta =$

the order of $\sigma_\alpha \sigma_\beta \big|_{\mathbb{R}\alpha \oplus \mathbb{R}\beta}$.

---

§11 classification.

$\Phi \subseteq E$, $\Delta \subseteq \Phi$ base

$\forall \Delta'$ base $\Rightarrow$ $\exists \sigma \in W$, $\sigma(\Delta') = \Delta$

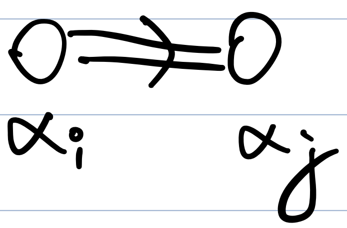
(Δ, C) recover Φ.

Def 11.8 (Dynkin diagram)

Coxeter graph + arrows

$\langle \alpha_i, \alpha_j \rangle \neq \langle \alpha_j, \alpha_i \rangle$

We can add an arrow pointing to the shorter root.

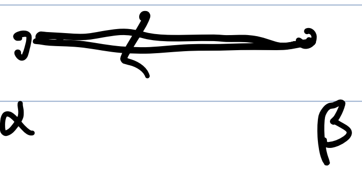


$\langle \alpha_i, \alpha_j \rangle = -2$ $\langle \alpha_j, \alpha_i \rangle = -1$

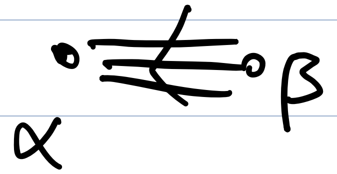
Dynkin diagram $\Leftrightarrow$ C $\Rightarrow$ Coxeter Diagram

(1)

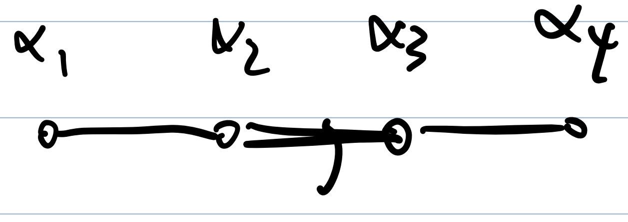
B_2



G_2



(2)



Dynkin diagram

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad F_4.$$

§ 11.3 irreducible components.

$\Phi \subseteq E$, $\exists \Delta \subseteq \Phi$ base → dynkin.

Remark. 11.10.

(Dynkin)

Φ irre. $\Leftrightarrow$ Coxeter graph

$\Updownarrow$ Connected

Δ irre.

Φ reducible $\Leftrightarrow$ ......

Prop 11.11

Φ decomposes (uniquely) as the union of irre. root systems

s.t. $E = E_1 \oplus \cdots \oplus E_t$

$\Phi = \sqcup \Phi_i$, $E_i = \mathrm{Span}\, \Phi_i$

pf: Suppose $\Delta = \underbrace{\Delta_1 \sqcup \cdots \sqcup \Delta_t}_{\left(\text{connected components of Coxeter graph} \right)}$

$E_i = \operatorname{Span}_{\mathbb{R}} \Delta_i$.

$\Phi_i = W(\Delta_i)$ $\quad$ W: Weyl gp of

Φ.

$$W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$$

$\alpha \in \Delta_i$, $\beta \in \Delta_j \Rightarrow \sigma_\alpha(\beta) = \beta$

$\Rightarrow \Phi_i \subseteq \operatorname{Span}_{\mathbb{Z}} \Delta_i$

Δ_i is the base of Φ_i.

Φ root system. $\Rightarrow$ $\Phi = \bigcup_{i=1}^{t} \Phi_i$

(uniquely):

$$\Phi = \Phi_1 \cup \cdots \cup \Phi_t$$

$$\Delta_1 \qquad \Delta_t$$

$= \Phi_1' \cup \cdots \cup \Phi_s'$

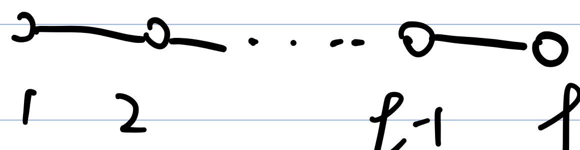
$\Delta_i' \triangleq \Delta \cap \Phi_i'$

$\Rightarrow$ Δ_i' : base of Φ_i', irre. ✓.

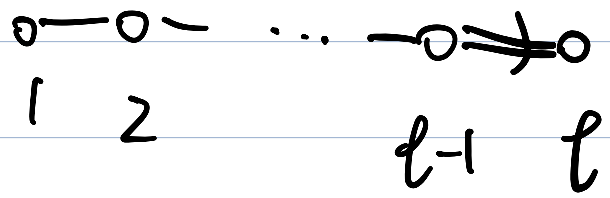
§11.4. Classification thm.

Theorem 11.12.

If Φ is an irr. root system of rank ℓ then its Dynkin diagram is one of the following:

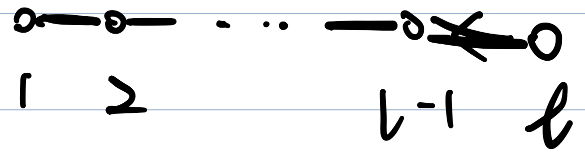
A_ℓ:  $\ell \geq 1$.

B_ℓ :



$\ell \geqslant 2$

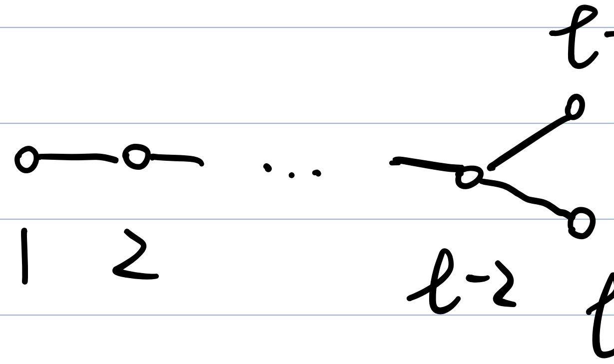
C_ℓ :



$\ell \geqslant 3.$

$(\Phi - B_\ell \qquad \Phi^\vee - C_\ell)$

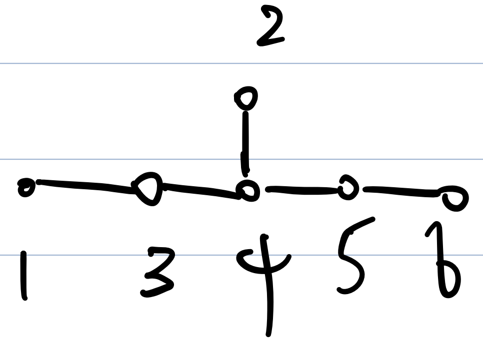
D_ℓ :



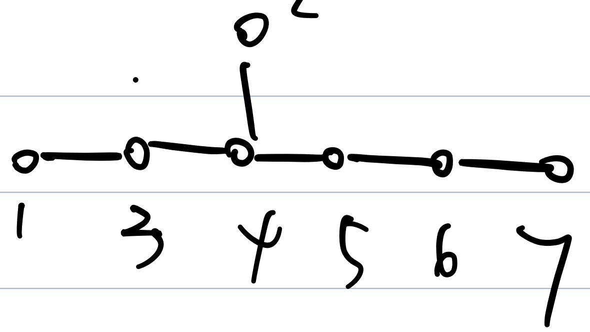
$\ell \geqslant 4$

$D_2 = A_1 \times A_1 \qquad D_3 = A_3$

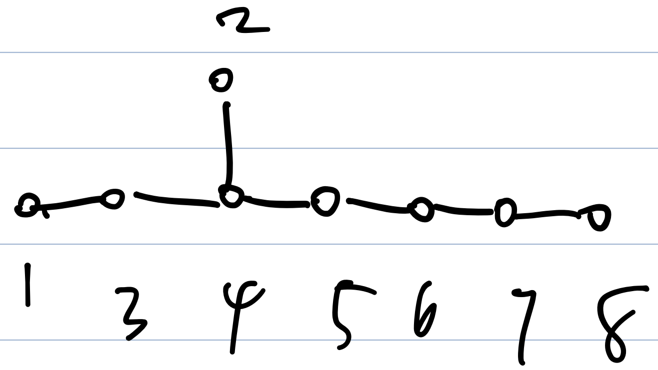
E_6 :



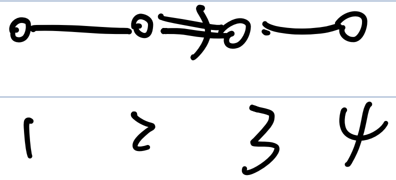
E7



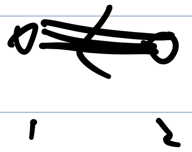
E 8



F4



G2



A, D, E types (single edges).

B, C, F, G

§ 12.1. Construction of type A - G

$\mathbb{R}^n$ $\varepsilon_1, \cdots, \varepsilon_n$ orthonormal basis

$$I = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$$

Theorem. 12.1. For each Dynkin diagram of type A - G

∃ an irre. root system

having the given diagram.

Pf: A_ℓ ($\ell \geq 1$).

sl_{l+1} $\quad E = \mathbb{R}^{n+1}$

$$E = P_{\varepsilon_1 + \cdots + \varepsilon_{n+1}}$$

$$\Phi = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \}.$$

$$\Delta = \{ \alpha_1, \cdots, \alpha_n \}$$

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$$

$\langle \alpha_i, \alpha_{i+1} \rangle = -1$, $\langle \alpha_i, \alpha_j \rangle = 0$, $|j-i| \geq 2$

$\exists A_n$.

$W = \langle \sigma_{\alpha_i} \mid 1 \leq i \leq \ell \rangle \leq GL(\mathbb{R}^{n+1})$.

$\mathbb{R}^{n+1} = E \oplus \mathbb{R}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$

$\sigma_{\alpha_i}\left(\sum_{k=1}^{\ell+1} a_k \varepsilon_k\right)$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$= \sum a_k \varepsilon_k - (a_i - a_{i+1})(\varepsilon_i - \varepsilon_{i+1})$

$= \sum_{k \neq i, i+1} a_k \varepsilon_k + a_{i+1} \varepsilon_i + a_i \varepsilon_{i+1}$

$a_i \longleftrightarrow a_{i+1}$

$\sigma_{\alpha_i} \in S_{\ell+1}$. $\sigma_{\alpha_i}(\varepsilon_i) = \varepsilon_{i+1}$

$\sigma_{\alpha_i}(\varepsilon_{i+1}) = \varepsilon_i$

$W = \langle (i, i+1) \rangle = S_{\ell+1}$.

B_ℓ $(\ell \geq 2)$ $E = \mathbb{R}^\ell$

$\Phi = \{ \alpha \in I \mid (\alpha, \alpha) = 1 \text{ or } 2 \}$.

$= \{ \pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \, (i \neq j) \}$.

$|\Phi| = 2l^2$

Φ root system

$\Delta = \{\alpha_1, \cdots, \alpha_l\}$

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \leq l-1.$

$\alpha_l = \varepsilon_l$

$$\varepsilon_i = \sum_{k=i}^{l} \alpha_k$$

$\alpha_i - \alpha_j \quad , i<j \quad \in \Phi^+$

$\alpha_i - \alpha_j \quad , \quad i>j \quad \in \Phi^-$

$$\alpha_i + \alpha_j \in \Phi^+$$

$$-(\alpha_i + \alpha_j) \in \Phi^-.$$

$$B_l = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -2 \\ 0 & & -1 & 2 \end{pmatrix}$$

$$\det B_l = 2 \det B_{l-1} - \det B_{l-2} = 2$$

$$W = \langle \sigma_{\alpha_i} \mid 1 \leq i \leq l \rangle$$

$$= \langle \sigma_{\alpha_i}, \sigma_{\varepsilon_j} \mid 1 \leq i \leq l-1, 1 \leq j \leq l \rangle$$

$$\begin{array}{c} W_1 = \langle \sigma_{\alpha_i} \mid 1 \le i \le l-1 \rangle \cong S_l. \\ \cap \\ W \end{array}$$

$$\begin{array}{c} W_2 = \langle \sigma_{\varepsilon_i} \mid 1 \le i \le l \rangle \subseteq W \\ \downarrow \cong \\ (\mathbb{Z}/2\mathbb{Z})^{l}. \end{array}$$

① Claim: $W_1 \cap W_2 = \{ \mathrm{Id}_E \}$

W_1, W_2 generates W.

$$\sigma_{\varepsilon_i - \varepsilon_j} \sigma_{\varepsilon_k} \sigma_{\varepsilon_i - \varepsilon_j} = \begin{cases} \sigma_{\varepsilon_k}, & k \notin \{i, j\} \end{cases}$$

$$\left| \begin{array}{ll} & k=i \\ \sigma_{\varepsilon_i} & k=j \end{array} \right.$$

$$\Rightarrow W_2 \triangleleft W.$$

$$\sigma_{\varepsilon_i+\varepsilon_j} = \sigma_{\varepsilon_j}\,\sigma_{\varepsilon_i-\varepsilon_j}\,\sigma_{\varepsilon_j}$$

$$\Rightarrow W_1 \ntriangleleft W$$

$$\Rightarrow W = W_1 \overset{\curvearrowright}{\ltimes} W_2 = S_l \overset{\curvearrowright}{\ltimes} (\mathbb{Z}/2\mathbb{Z})^l$$

$$C_l \quad (l \geq 3) \qquad \Phi \to \Phi^\vee$$

$$W_\Phi \xrightarrow{\sim} W_{\Phi^\vee}$$

$A \subseteq \Phi \implies A^\vee \subseteq \Phi^\vee$ ✓.

$\|$

$\underset{\sim}{A(\gamma)}$

D_l. $\quad E = \mathbb{R}^l$

$$\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\}$$

$$= \{\pm \varepsilon_i \pm \varepsilon_j\}$$

$$|\Phi| = l^2 - 2l$$

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \le i \le l-1$$

$$\alpha_l = \varepsilon_{l-1} + \varepsilon_l$$

$$\varepsilon_i - \varepsilon_j = \sum_{k=i}^{j-1} \alpha_k \in \Phi^+ \quad i<j$$

$$\varepsilon_i + \varepsilon_j = \varepsilon_i - \varepsilon_j + 2\varepsilon_j \in \Phi^+$$

$$\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$$

$$W = \langle \sigma_{\varepsilon_i - \varepsilon_j}, \sigma_{\varepsilon_i + \varepsilon_j} \mid i<j \rangle$$

$$W_1 = \langle \sigma_{\varepsilon_i - \varepsilon_j} \mid i<j \rangle$$

$$\sigma_{\varepsilon_i + \varepsilon_j}(a\varepsilon_i + b\varepsilon_j) = -a\varepsilon_j - b\varepsilon_i$$

$$\begin{pmatrix} 0 & -b \\ -a & 0 \end{pmatrix}$$

$$W_2 = \langle \underset{\substack{\shortparallel \\ \sigma_i}}{\sigma_{\varepsilon_i-\varepsilon_j}\sigma_{\varepsilon_i+\varepsilon_j}} \mid i<j \rangle$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

$$\Rightarrow W_2 = (\mathbb{Z}/2\mathbb{Z})^{\ell-1}$$

$$W_2 \lhd W$$

$$W = \overset{\curvearrowright}{W_1 \ltimes W_2} = S_\ell \ltimes \underset{\curvearrowright}{(\mathbb{Z}/2\mathbb{Z})^{\ell-1}}$$

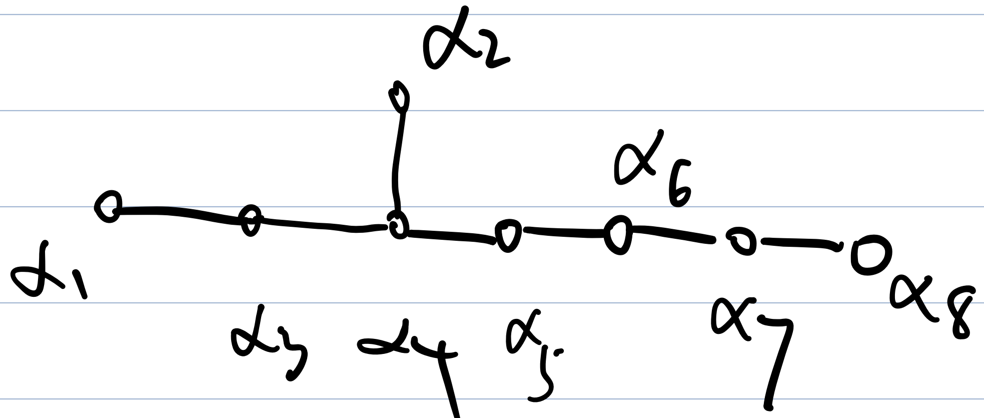
Cartan matrix

$\shortparallel$

$$\begin{pmatrix} 2 & -1 & \cdots & & & \\ -1 & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

$l \geq 3.$

$E_{6,7,8}$



Φ_{E_8} $\qquad$ $\Delta_{E_8} = \{\alpha_1, \cdots, \alpha_8\}$

$\Phi_{E_7} \triangleq \Phi_{E_8} \cap \text{Span}\{\alpha_1, \cdots, \alpha_7\}$

$\Phi_{E_6} = \cdots\cdots\cdots$

$E = \mathbb{R}^8 \qquad I' \triangleq I + \mathbb{Z}\cdot\frac{\varepsilon_1+\dots+\varepsilon_8}{2}$

$$I'' = \left\{ \sum_{i=1}^{8} c_i\varepsilon_i + c\,\frac{\varepsilon_1+\dots+\varepsilon_8}{2} \;\middle|\; \begin{array}{l} \sum_{i=1}^{8} c_i \in 2\mathbb{Z},\ c_i \in \mathbb{Z} \\ c = 0 \text{ or } 1 \end{array} \right\}$$

$I'' \subseteq I'$

$$\Phi = \{ \alpha \in I'' \mid \|\alpha\|^2 = 2 \}$$

$$= \left\{ \sum_{i=1}^{8} \left(c_i + \frac{c}{2}\right)\varepsilon_i \;\middle|\; \sum_{i=1}^{8} (c_i^2 + cc_i) + 2c^2 = 2 \right\}$$

$C=0 \quad \pm\varepsilon_i \pm \varepsilon_j \quad i \neq j. \quad 112.$

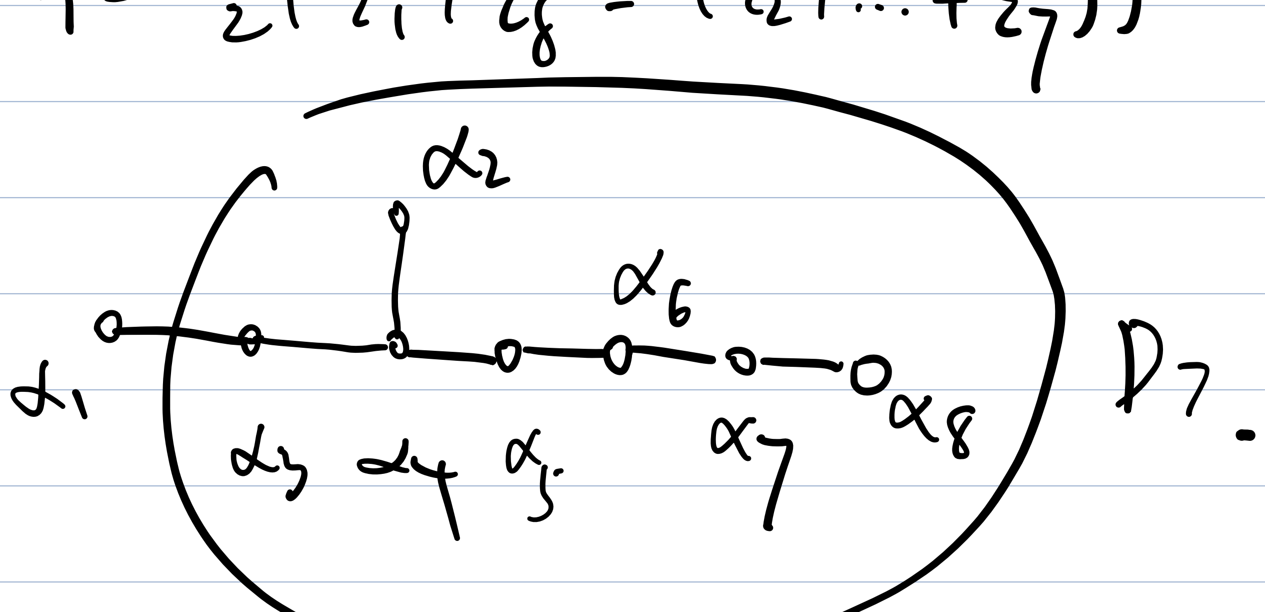
$C=1 \quad \{c_i + \frac{1}{2}\} \subseteq \{\pm\frac{1}{2}\}$

$$\alpha = \frac{1}{2}\sum_{i=1}^{8}(-1)^{k(i)}\varepsilon_i$$

$\frac{1}{2}\sum(-1)^{k(i)} \in 2\mathbb{Z}$

$112 + 2 + 28 + 70 + 28 = 240.$

$\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - (\varepsilon_2 + \dots + \varepsilon_7))$



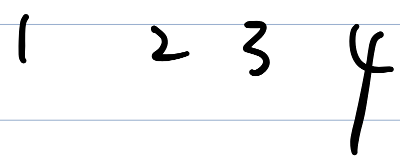
$$\left(\begin{array}{c|ccc} 2 & 0 & -1 & \\ \hline 0 & & & \\ -1 & & D_7, & \\ & & & \end{array}\right)$$

The order of Weyl gp

$$\| $$

$$2^{14} 3^{5} 5^{2} 7$$

$F_4.$



$E = \mathbb{R}^4$ $\quad I' = I + \mathbb{Z}\frac{\varepsilon}{2}$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$$

$$\Phi = \{ \alpha \in I' \mid \|\alpha\| = \{1, 2\} \}$$

$\pm\varepsilon_i, \ \pm\varepsilon_i \pm \varepsilon_j$ 32.

$\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ 16. $\Rightarrow 48$

Dim of Lie algebra of $F_4 = 52$.

$\alpha_1 = \varepsilon_2 - \varepsilon_3$

$\alpha_2 = \varepsilon_3 - \varepsilon_4$

$\alpha_3 = \varepsilon_4$

$\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$

$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} \qquad \det = 1.$$

$|W| = 1152$

$\theta_\ell = \varepsilon_1 + \varepsilon_2$

$\theta_s = \varepsilon_1$

$G_2 \qquad E = P_\varepsilon \subseteq \mathbb{R}^3 \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$I' = I \cap E$

$= \{ \sum a_i \varepsilon_i \mid \sum a_i = 0,\ a_i \in \mathbb{Z} \}$

$$\Phi = \{\alpha \in I' \mid \|\alpha\| = 2 \text{ or } 6\}$$

$\|\alpha\|^2 = 2 \Rightarrow \varepsilon_i - \varepsilon_j, \quad i \neq j.$ 6.

$\|\alpha\|^2 = 6 = 2^2 + 1 + 1$

$\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k)$ 6.

$j < k$

$W = D_6.$

---

§ 12.2 Aut(Φ).

$$\mathrm{Aut}(\Phi) = \{\sigma \in GL(E) \mid \sigma(\Phi) = \Phi\}$$

$$\Downarrow$$

Lemma 9.1 $\begin{array}{c}\langle\sigma(\alpha), \sigma(\beta)\rangle \\ \| \\ \langle\alpha, \beta\rangle\end{array}$

① $W \subseteq \mathrm{Aut}(\Phi)$.

$\forall\, \sigma \in \mathrm{Aut}(\Phi)$. $\sigma(\Phi) = \Phi$.

$$\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)} \in W$$

$\Longrightarrow$ $W \trianglelefteq \mathrm{Aut}(\Phi)$.

Fix $\Delta \subseteq \Phi$ base.

② $\gamma \triangleq \{\sigma \in \mathrm{Aut}(\Phi) \mid \sigma(\Delta) = \Delta\}$

$< \mathrm{Aut}(\Phi)$.

$\sigma \in \gamma \cap \mathcal{U}$

Theorem 10.13 $\Rightarrow$ $\sigma = \mathrm{Id}_E$

$\Rightarrow \gamma \cap \mathcal{U} = \{\mathrm{Id}_E\}$

Prop 12.2. $\mathrm{Aut}(\Phi) = \gamma \ltimes \mathcal{U}$

Pf: $\forall \tau \in \mathrm{Aut}(\Phi)$

$\Delta \subseteq \Phi \Rightarrow \tau(\Delta) \subseteq \Phi$ base

By theorem 10.13, $\exists \sigma \in W$

$$\sigma\tau(\Delta) = \Delta$$

$$\Rightarrow \sigma\tau \in \gamma$$

$$\Rightarrow \tau \in \gamma W = W\gamma$$

$$\Rightarrow \mathrm{Aut}(\Phi) = \gamma \ltimes W$$

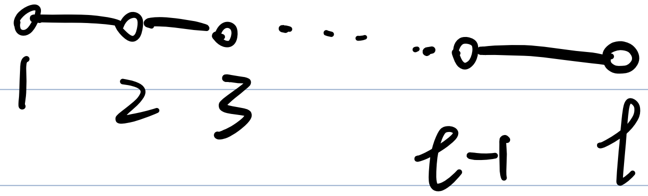
开口向正规子群.

$\sigma \in \gamma$, auto. of Dynkin diagram.

$$\sigma(\Delta) = \Delta \qquad \begin{array}{c} \langle \sigma(\alpha), \sigma(\beta) \rangle \\ \shortparallel \\ \langle \alpha, \beta \rangle \end{array}$$

Φ irre.

A_ℓ

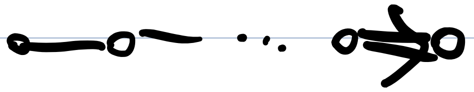


$\sigma \in \gamma \Leftrightarrow \sigma = Id$

$\sigma : i \mapsto n+1-i$

$\mathbb{Z}/2\mathbb{Z}$.

B_ℓ .



$\gamma = \{Id\}$

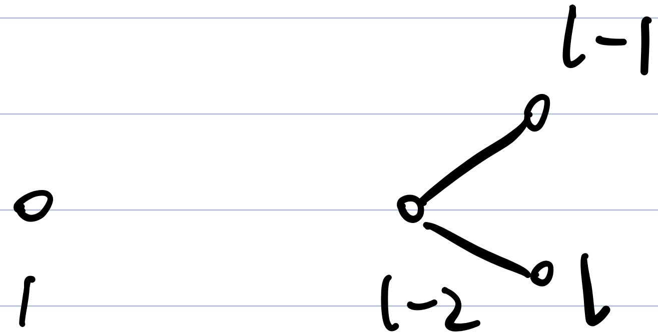
C_ℓ - · -

$\gamma = \{Id\}$

F_4, G_2

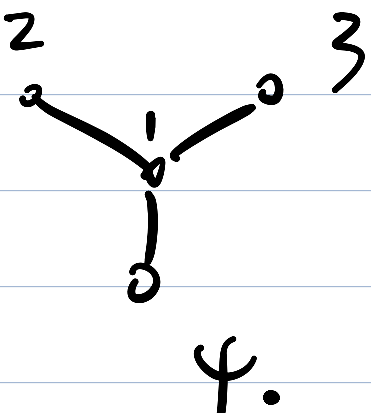
$\gamma = \{Id\}$

D_ℓ :

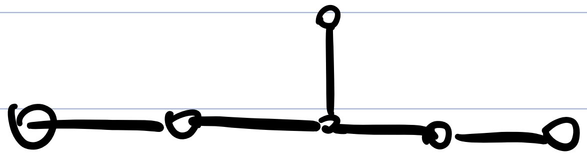


$l > 4$: Id or $l-1 \leftrightarrow l$ $\mathbb{Z}/2\mathbb{Z}$

$l = 4$:



S_3.

E_6  $\mathbb{Z}/2\mathbb{Z}$

E_7, E_8 $\{\mathrm{Id}\}$

When $-\mathrm{Id}_E$ is in W?

L s.s. $H \subseteq L$ maximal toral

$$L = H \oplus \sum_{\alpha \in \Phi} L_\alpha \qquad \dim L_\alpha = 1$$

$$L \quad \begin{matrix} H_1 \\ H_2 \end{matrix} \Longrightarrow \begin{matrix} \Phi_1 \\ \Phi_2 . \end{matrix}$$

$$H \subseteq L \longrightarrow \Phi .$$

$$\tau \in \mathrm{Aut}(\Phi) \rightsquigarrow \tilde{\tau} \in \mathrm{Aut}(\tilde{L}) .$$

§13. Abstract theory of weights.

§13.1 Weights.

Def 13.1. $\Phi \subseteq E$ root system

$\lambda \in E$ is called a weight if

$\langle \lambda, \alpha \rangle \in \mathbb{Z}, \quad \forall \alpha \in \Phi$

$\Lambda = \{$all weights$\}$

$$\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

$\Rightarrow \Lambda \subseteq E$ subgroup

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

If $\Delta \subseteq \Phi$ base

$\Rightarrow$ $\Delta^{\vee} \subseteq \Phi$ base

$$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$$

$$(\lambda, \alpha^{\vee}) \in \mathbb{Z}, \forall \alpha \in \Phi$$

$$\Updownarrow \qquad ;$$

$$\langle \lambda, \alpha \rangle \in \mathbb{Z}$$

Hence $\langle \lambda, \alpha \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi$

$$\Updownarrow$$

$$\langle \lambda, \alpha^{\vee} \rangle \in \mathbb{Z}$$

Claim: $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$

$\Rightarrow \forall \sigma \in W$

$$\sigma(\lambda) - \lambda \in \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$$

Pf:

$$\sigma_{\alpha_k}(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha_k \rangle}_{\uparrow \, \mathbb{Z}} \alpha_k$$

$\Delta \subseteq \Phi$ base $\qquad \Delta^\vee \subseteq \Phi^\vee$ base

$\Rightarrow \Delta^\vee$ is a basis of E

$\lambda \in \Lambda \iff (\lambda, \alpha_i^\vee) \in E$

Denote $\{\lambda_1, \dots, \lambda_k\}$ the dual basis of $\{\alpha_1^\vee, \dots, \alpha_k^\vee\}$

Define $\Lambda_r = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$

$$= \bigoplus_{\alpha \in \mathbb{Z}} \mathbb{Z}\alpha$$

$\Rightarrow \Lambda_r \subseteq \Lambda \subseteq E$

Λ_r: root lattice

Λ: weight lattice

Def 13.2. fix $\Delta \subseteq E$

(1) $\lambda \in \Lambda$ is called dominant

$\iff \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$, $\forall \alpha \in \Delta$

$\Lambda^+ = \Lambda \cap \overline{C(\Delta)}$

(2) $\lambda \in \Lambda$ is called simply dominant

$\iff \langle \lambda, \alpha \rangle \in \mathbb{Z}_{>0}$, $\forall \alpha \in \Delta$

$\Lambda^{++} = \Lambda \cap C(\Delta)$

Def 13.3.

the dual basis $\{\lambda_1, \cdots, \lambda_\ell\}$

of $\{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\}$ are called

the fundamental dominant weight.

If $\lambda = \sum_{i=1}^{\ell} m_i \lambda_i \in E$ $(\lambda_i, \alpha_j^\vee)$

$$\langle \lambda, \alpha_j \rangle = m_j$$

$$\lambda \in \Lambda \iff \lambda \in \oplus \mathbb{Z} \lambda_i$$

$$\Rightarrow \quad \Lambda = \bigoplus_{i=1}^{\ell} \mathbb{Z} \lambda_i$$

Assume $\alpha_i = \sum_{j=1}^{\ell} m_{ij} \lambda_j$

$$= \sum_{j=1}^{\ell} \langle \alpha_i, \alpha_j \rangle \lambda_j$$

$$(\alpha_1, \cdots, \alpha_\ell) = (\lambda_1, \cdots, \lambda_\ell) \; C^T$$

$$\Rightarrow |\Lambda/\Lambda_r| = \det C$$

A_2. $$(\alpha_1 \quad \alpha_2) = (\lambda_1 \quad \lambda_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Order of $\Lambda/\Lambda_r = \det C$.

---

§ 13.2 Dominant weights.

$$\Lambda^{+} = \Lambda \cap \overline{C(A)} = \bigoplus_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \lambda_i$$

$\forall \sigma \in W \quad \lambda \in \Lambda \quad \sigma(\lambda) \in \Lambda$

Lemma 13.4. (1) $\forall \lambda \in \Lambda, \exists \sigma \in W$

$\sigma(\lambda) \in \Lambda^{+}$

If $\sigma_1(\lambda), \sigma_2(\lambda) \in \Lambda^{+}$

$\Rightarrow \sigma_1(\lambda) = \sigma_2(\lambda)$

$(|W(\lambda) \cap \Lambda^{+}| = 1)$

2) If $\lambda \in \Lambda^{+}$, then

$$\lambda - \sigma(\lambda) \in \bigoplus_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\, \alpha_i$$

i.e. $\sigma(\lambda) \prec \lambda$

(3) If $\lambda \in \Lambda^{++}$

$$\sigma(\lambda) = \lambda \Leftrightarrow \sigma = Id_E$$

Pf: (1) Lemma 10.16

(2) $\lambda \in \Lambda^{+} \Rightarrow \lambda = \sum_{i=1}^{\ell} m_i \lambda_i$

$m_i \geq 0$

$\ell(\sigma) = t$ $\quad \sigma = \sigma_{\alpha_{i_1}} \cdots \sigma_{\alpha_{i_t}}$

$\alpha_{i_k} \in A$

Example 13.5. $\alpha_1 = 2\lambda_1 - \lambda_2$

$\alpha_2 = 2\lambda_2 - \lambda_1$

$\alpha_1 = 2\lambda_1 - \lambda_2$

$\alpha_2 = 2\lambda_2 - \lambda_1$

$\lambda_1 \in \Lambda^+$, $\lambda_1 + \alpha_1 > \lambda_1$

$\lambda_1 + \alpha_1 \notin \Lambda^+$

Lemma 13.6 let $\lambda \in \Lambda^+$

then $\{\mu \in \Lambda^+ \mid \mu < \lambda\}$ is finite

Pf: $\lambda - \mu = \sum_{i=1}^{l} n_i \alpha_i$, $n_i \geq 0$.

$$(\lambda, \lambda) = (\lambda - \mu, \lambda) + (\mu, \lambda)$$

$$= (\lambda - \mu, \lambda) + (\mu, \lambda - \mu) + (\mu, \mu)$$

$$\geq (\mu, \mu)$$

$\Rightarrow \{\mu \in \Lambda^+ \mid \mu < \lambda\} \subseteq \Lambda^+ \cap B(\|\lambda\|)$.

□

§ 13.3. The weight $\delta = \frac{1}{2}\sum_{\alpha\in\Phi} \alpha$

$\Delta \subseteq \Phi \quad \langle \delta, \alpha \rangle = 1, \ \forall \alpha \in \Delta$

$(\sigma_i \delta = \delta - \alpha_i)$

$\Rightarrow \delta = \sum_{i=1}^{\ell} \lambda_i \in \Lambda^{++}$

Lemma 13.7. $\delta = \sum \lambda_i \in \Lambda^{++}$

Lemma 13.8. $\mu \in \Lambda^{+}$, $\nu = \sigma(\mu)$,

$\sigma \in W$

Then $(\nu+\delta, \nu+\delta) \leq (\mu+\delta, \mu+\delta)$

"=" $\Leftrightarrow \nu = \mu$

§13.4. Saturated sets of weights

Def 13.4. Φ, Λ

(1) $\Pi \subseteq \Lambda$ is saturated if

$\forall \lambda \in \Pi$, $\alpha \in \Phi$, any i between

0 and $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ (irre. repn.)

$$\lambda - i\alpha \in \Pi$$

(2) We say a saturated Π

has highest weight λ if

$\lambda \in \Pi$, $\forall \mu \in \Pi$, $\mu \prec \lambda$

(*) If Π has highest weight λ,

then $\lambda \in \Lambda^+$, $\forall \lambda \in \Pi$, $\sigma \in W$

$\Rightarrow \sigma(\lambda) \in \Pi$

$\exists \sigma_1$, $\sigma_1(\lambda) \in \Lambda^+$

$\sigma_1(\lambda) \prec \lambda$

\(**) Π saturated $\Rightarrow \sigma(\Pi) = \Pi$

Example 13.10.

(1) $\Pi = \Phi \cup \{0\}$ saturated.

(2) If Φ is irre.

$\Pi = \Phi \cup \{0\}$ has highest weight.

Lemma 13.11

If Π is a saturated set

with h.w λ

$\Rightarrow$ $|\Pi| < +\infty$

Pf: $\Pi \subseteq W(\underbrace{\Pi \cap \Lambda^+}_{\text{finite.}})$

Lemma 13.12.

If Π is saturated with h.w. λ, then $\mu \in \Lambda^+$, $\mu < \lambda$

$\Rightarrow \mu \in \Pi$

Pf: Suppose $\mu' = \mu + \sum_{\alpha \in \Delta} k_\alpha \alpha \in \Pi$

$k_\alpha \in \mathbb{Z}_{\geq 0}$

$\mu' \neq \mu$

$$\Rightarrow \quad \left(\sum_{\alpha} k_\alpha \alpha, \sum_{\alpha} k_\alpha \alpha\right) > 0$$

$\Rightarrow \exists\, \beta \in A$ s.t.

$$\left(\sum k_\alpha \alpha, \beta\right) > 0,$$

$$\Rightarrow \langle \mu', \beta \rangle > 0 \quad \mu' \in \Pi$$

$$\Rightarrow \langle \mu', \beta \rangle \geq 1$$

$\Rightarrow \mu' - \beta \in \Pi$ (saturated condition).

$$\Rightarrow \mu + \sum_{\alpha \neq \beta} k_\alpha \alpha + (k_\beta - 1)\beta \in \Pi$$

Induction on $\sum_{\alpha \in \Delta} k_\alpha$

$$\Rightarrow \mu \in \Pi$$

Remark. ①

If Π is saturated with

h.w. $\lambda \in \Lambda^+$

$\Pi_1 = \{ \mu \in \Lambda^+ \mid \mu < \lambda \}$

$\Rightarrow \quad \Pi = W(\Pi_1)$

② $\Pi_1 = \{ \mu \in \Lambda^+ \mid \mu < \lambda \}$

$\Pi = W(\Pi_1)$

$\Rightarrow \Pi$ saturated (Ex. 10)

Lemma. 13.13.

Π saturated with h.w. λ

$\mu \in \Pi$

$\Rightarrow (\mu+\delta, \mu+\delta) \leq (\lambda+\delta, \lambda+\delta)$

Pf:

$$(\lambda, \lambda) - (\mu, \mu) = (\lambda-\mu, \lambda) + (\mu, \lambda-\mu)$$

$$\geq 0$$

$\forall \nu \in \Lambda \Rightarrow \exists \sigma \in W$

$\sigma(\nu) = \mu \in \Lambda^+, \quad \mu > \nu$

$(\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta)$

$= (\lambda, \lambda) - (\mu, \mu) + 2(\lambda-\mu, \delta)$

≥ 0

"$=$" $\Longleftrightarrow$ $\lambda = \mu$

$$\begin{tikzcd}
\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_t \arrow[d, "?"] \\
E = E_1 \oplus \cdots \oplus E_t
\end{tikzcd}$$

Chapter IV. Isomorphism and Conjugacy thm.

§14. Isomorphism.

§14.1. Reduction to the

Simple case

Prop 14.1.

Let L be a simple Lie, H, Φ, then Φ is irre.

Pf: If $\Phi = \Phi_1 \sqcup \Phi_2$

$(\Phi_1, \Phi_2) = 0$, $\Phi_i \neq \phi$

$\forall \alpha \in \Phi_1$ $\beta \in \Phi_2$

$\Rightarrow \quad \alpha+\beta \notin \Phi$

$\because (\alpha+\beta, \beta) \neq 0, \quad (\alpha+\beta, \alpha) \neq 0$

$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} = 0$

$L_1 \triangleq \langle L_\alpha \mid \alpha \in \Phi_1 \rangle$

$\forall \alpha \in \Phi \quad [L_\alpha, L_1] \subseteq L_1$

$L_1 \triangleleft L$

$\Rightarrow L_1 = L$, $\wedge$

Cor 14.2.

If L s.s. H maximal

toral $\rightarrow \Phi$

$L = L_1 \oplus \cdots \oplus L_t$

L_i simple $\quad H_i = H \cap L_i$

$\Rightarrow H = H_1 \oplus \cdots \oplus H_t$

Then $H_i \subseteq L_i$ is a maximal

toral subalg. ✓.

$\Phi_i \subseteq H_i^*$ root system.

$\Rightarrow$ Φ_i irr. ✓.

$$\Phi = \underbrace{\Phi_1 \cup \cdots \cup \Phi_t}_{\text{Nontrivial part}}$$

pf:

① $H_1 \oplus \cdots \oplus H_t \geq H$

$\forall h \in H$

$\Rightarrow h = h_1 + \cdots + h_t \quad h_i \in L_i$

For $\forall h' \in H$, $h' = \sum h_i'$

$[h, h'] = 0 \Rightarrow [h_i, h_i'] = 0$

$\Rightarrow [h_i, h'] = 0$

$\Rightarrow h_i \in C_L(H) = H$

② $H_i \subseteq L_i$ toral ✓.

③ $\Phi = \bigcup \Phi_i$

$$\mathcal{L}_i = H_i \oplus \sum_{\alpha \in \Phi_i} \underset{\sim}{(\mathcal{L}_i)_\alpha}$$

$$H_i \cdot (\mathcal{L}_i)_\alpha \subseteq (\mathcal{L}_i)_\alpha$$

$j \neq i \quad H_j \, (\mathcal{L}_i)_\alpha = 0$

Extend α by $\tilde{\alpha}(H_j) = 0$.

$\forall j \neq i$

$\Rightarrow \tilde{\alpha} \in \Phi$.

$\Rightarrow \Phi \supseteq \Phi_1 \cup \cdots \cup \Phi_t$

Count dim

$\Rightarrow \quad \Phi = \Phi_1 \cup \dots \cup \Phi_t.$

§14.2 Isomorphism thm.

Prop 14.3 L s.s.

$H \subseteq L \longrightarrow \Phi$

If $\Delta \subseteq \Phi$ base

$\Rightarrow \quad L = \langle L_\alpha, L_{-\alpha} \mid \alpha \in \Delta \rangle$

$(\mathcal{L}, H) \sim \Phi$

$(\mathcal{L}', H') \sim \Phi'$ $\quad\quad \Phi \xrightarrow[\sim]{\psi} \Phi'$

Claim: $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$

$\pi : H \longrightarrow H'$ $\quad\quad \kappa(t_\alpha, h) = \alpha(h)$
$\forall h \in H$

$t_\alpha \longrightarrow t_{\psi(\alpha)}$

$H \to H^*$
$\alpha \to t_\alpha$ $\quad\quad$ iso of vector space.

Theorem 14.4

$$\pi : H \to H'$$

$\{\alpha_1, \ldots, \alpha_k\}$ basis.

$$t_\alpha \to t'_{\psi(\alpha)}$$

$\Rightarrow \forall \beta \in \Phi \quad t_\beta \to t_{\psi(\beta)}$

Fix $\Delta \subseteq \Phi$, $\Delta' = \{\psi(\alpha) \mid \alpha \in \Delta\} \subseteq \Phi'$

base. For $\forall \alpha \in \Delta$, choose $0 \neq x_\alpha \in L_\alpha$

Then extends to a unique iso.

$\pi : L \to L'$ extending $\pi : H \to H'$

and $\pi(x_\alpha) = x'_{\psi(\alpha)}$

pf: $[x_\alpha, y_\alpha] = h_\alpha = \dfrac{2t_\alpha}{(\alpha,\alpha)} \mapsto h'_{\psi(\alpha)}$

"$(\alpha,\alpha) = (\alpha', \alpha')$"

$\Rightarrow \pi([x_\alpha, y_\alpha]) = [\pi(x_\alpha), \pi(y_\alpha)]$

$= [x_{\psi'(\alpha)}, \pi(y_\alpha)]$

$\Rightarrow \pi(y_\alpha) = y'_{\psi(\alpha)}$

① Define $L'' = L \oplus L'$ s.s.

$x_\alpha \in L_\alpha$ $y_\alpha \in L_{-\alpha}$

$x'_{\varphi\alpha}$ $y'_{\varphi(\alpha)}$

Define $\bar{x}_\alpha = (x_\alpha, x'_{\varphi(\alpha)}) \in L''$

$\bar{y}_\alpha = (y_\alpha, y'_{\varphi(\alpha)})$

$D \triangleq \langle \bar{x}_\alpha, \bar{y}_\alpha \rangle$

② L simple $\Rightarrow$ Φ irr.

$\Rightarrow$ $\exists$ highest root β

L' simple $\Rightarrow$ Φ' irr. see text book.

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§14.3 $\mathrm{Aut}(L)$

Prop 14.5. L, H, Φ, Δ

Fix $0 \neq x_\alpha \in L_\alpha$, $\alpha \in \Delta$

$y_\alpha \in L_{-\alpha}$ $[x_\alpha, y_\alpha] = h_\alpha$

Then there exists $\sigma \in \mathrm{Aut}(Z)$

$\sigma(x_\alpha) = -y_\alpha$, $\sigma(y_\alpha) = -x_\alpha$

$\forall \alpha \in \Delta$ $\sigma(h) = -h$

$\sigma^2 = \mathrm{Id}\, E$

Pf: $\gamma : \Phi \to \Phi$

$\alpha \to -\alpha$

§ 16. Conjugacy thm.

$\bar{F} = F$, $\mathrm{char}\, F = 0$

Cartan Subalgebra

L s.s. $\qquad H \subseteq L$ CSA

$\Updownarrow$

maximal toral

§ 16.1. The group $\mathcal{E}(L)$

L, $\quad x \in L$

$$L = L_0(\mathrm{ad}\,x) \oplus \sum_{a \neq 0} L_a(\mathrm{ad}\,x)$$

Def 16.1.

Let L be a Lie algebra.

Call $x \in L$ strongly ad-nilpotent

$\Leftrightarrow \exists\, y \in L, a \neq 0$, s.t.

$$x \in L_a(\mathrm{ad}\, y)$$

$$[L_a(\mathrm{ad}\, y), L_b(\mathrm{ad}\, y)] \subseteq L_{a+b}(\mathrm{ad}\, y)$$

$+ \quad x$ ad- nilp.

Denote $\mathcal{N}(L) = \{x \in L \mid x$ is strongly ad-nil.$\}$

$$\mathcal{E}(L) ::= \langle e^{\mathrm{ad}\,x} \mid x \in \mathcal{N}(L) \rangle < \mathrm{Int}(L).$$

Remark 16.2.

(1) $\mathcal{E}(L) \lhd \mathrm{Aut}(L)$ $\forall \varphi \in \mathrm{Aut}(L)$

$x \in \mathcal{N}(L)$.

$\Rightarrow$ $a \neq 0$, $y \in L$ $\quad x \in L_a(\mathrm{ad}\, y)$

$\Rightarrow (\mathrm{ad}\,y - a)^k (x) = 0$

$\Rightarrow (\mathrm{ad}\,\varphi(y) - a)^k (\varphi(x)) =$

$\Rightarrow \varphi(x) \in L_a(\mathrm{ad}\,\varphi(y))$.

$\varphi e^{\mathrm{ad}x} \varphi^{-1} = e^{\mathrm{ad}\,\varphi(x)}$.

(2) $K \subseteq L \Rightarrow N(K) \subseteq N(L)$

But $\mathrm{ad}_K x$ nilp. $\not\Rightarrow$ $\mathrm{ad}_L x$ nilp.

Lemma 16.3. If $\varphi: L \to L'$ is epimorphism, $y \in L$

then $\underline{\varphi(L_a(\operatorname{ad} y)) = L'_a(\operatorname{ad} \varphi(y))}$.

$\Rightarrow$ $\varphi(N(L)) = N'(L)$.

Lemma 16.4. Let $\varphi: L \to L'$

be an epimorphism. If $\sigma' \in \mathcal{E}(L')$

then

$\exists\ \sigma \in \mathcal{E}(L)$ s.t.

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[d, "\exists\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & L' \arrow[d, "\sigma'"] \\
L \arrow[r, "\varphi"] & L'
\end{tikzcd}$$

Pf: $\forall x' \in N(L')$ $\quad \sigma' = e^{ad_{L'} x'}$

16.3 $\Rightarrow$ $\exists x, \quad \varphi(x) = x'$

$\sigma = e^{ad_L x}$

$\forall z \in L$

$ad_{L'} \varphi(x)$

$\varphi (e^{\mathrm{ad} z} (z)) = e (\varphi(z))$

§ 16.2. Conjugacy of CSAs

(solvable cases).

Thm 16.5.

Let L be solvable, then

$\forall\ H_1, H_2 \subseteq L$ are CSAs

$\exists\ \sigma \in \mathcal{E}(L)$ s.t.

$\sigma(H_1) = H_2$

§ 16.2

Theorem 16.5

L solvable

$\Rightarrow$ If $H_1, H_2 \subset L$ CSA

then $\exists\ \sigma \in \mathcal{E}(L)$, $\sigma(H_1) = H_2$

§ 16.3 Borel subalgebra

Define 16.6

Borel subalgebra $\Leftrightarrow$ maximal

solvable subalgebra

$K < L$ nil $\Rightarrow$ K solvable

$\forall H$ CSA, $\exists B$ Borel subalg.

s.t. $H \subseteq B$

Lemma 16.7

If B is a Borel subalg.

$\Rightarrow N_L(B) = B$

Pf: $N_L(B) = \{x \in L \mid [x, B] \subseteq B\}$

$\forall x \in N_L(B)$, $B' = B + \mathbb{F}x$

$\Rightarrow [B', B'] \subseteq B$

$\Rightarrow B'$ solvable

$\Rightarrow B = B'$

Lemma. 16.8.

If $\operatorname{Rad} L \neq L$

$\Rightarrow \{\text{Borel alg of } L\}$

$\uparrow$ 1:1

$\downarrow$

$\{$ Borel of $L/\operatorname{Rad} L \}$

Pf: $B < L$ Borel

$\Rightarrow$ $B + \operatorname{Rad} L / \operatorname{Rad} L$ solvable

---

L s.s. $\quad H \leq L$ CSA

$H < L$ CSA

Set

$N(\Delta) = \sum_{\alpha \in \Phi^+} L_\alpha$

$$B(\Delta) = H \oplus N(\Delta)$$

$\Rightarrow$ $N(\Delta)$ nilpotent

Lemma 16.10.

L s.s. $H < L$ CSA

Φ

① $\Delta \subseteq \Phi$, $B(\Delta)$ is a Borel subalg

② If $\Delta, \Delta' \subseteq \Phi$

$\Rightarrow \exists \sigma \in \mathcal{E}(L), \quad \sigma(B(\Delta))$

$= B(\Delta')$

Pf: See Humphreys.

§ 16.4. Conjugacy of Borel subalgebr

Theorem 16.9.

$B, B' < L$ Borel subalg.

$\Rightarrow \exists\, \sigma \in \mathcal{E}(L),\ \sigma(B) = B'$

Cor 16.10 $H_1, H_2 < L$ CSA

$\Rightarrow \exists\ \sigma \in \mathcal{E}(L)$ s.t. $\sigma(H_1) = H_2$

Pf: $H_1 \subseteq B_1$

$H_2 \subseteq B_2$

$\exists\ \sigma_1 \in \mathcal{E}(L),\ \sigma_1(B_1) = B_2$

B_2 solvable

$\mathcal{E}(L, B_2)|_{B_2}$

$\cap$

$\Rightarrow \exists \quad \in \mathcal{E}(B) \subseteq \mathcal{E}(L)$

$\Rightarrow \exists\, \sigma_2 \in \Sigma(B_2) \subseteq \Sigma(L)$

$\sigma_2(\sigma_1(H_1)) = H_2$

Let $\sigma = \sigma_2 \circ \sigma_1$

$\square$

this is why

we introduce "$\Sigma(L)$"

§ 16.5 $\mathrm{Aut}(L) > \mathrm{Int}(L) > \Sigma(L)$

$\mathrm{Aut}(\Phi) \cong \Gamma \ltimes W$

Remark. $\mathrm{Aut}(\mathcal{L}) = \gamma(\mathcal{L}) \ltimes \mathrm{Int}(\mathcal{L})$

$$\mathrm{Int}(\mathcal{L}) = \mathcal{E}(\mathcal{L})$$